QUANTITY AS LIMIT

Leibniz on the Metaphysics of Quantity[*]

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Abstract

This paper deals with the metaphysics of the notion of quantity in the philosophy of Leibniz, and its aim is to defend the following bi-conditional: for any object x, x has a certain quantity if and only if x has a (metaphysical) limit or a bound. The direction from left to right is justified in §3, while in §4 I develop an argument to justify the direction from right to left. Since the bi-conditional links the metaphysical notion of limit to the mathematical notion of quantity (and in this way it links Leibniz's metaphysics with his conception of *Mathesis Universalis*), it allows the use of metaphysics to clarify the features of the mathematical notion of quantity. This task is accomplished in §5 and §6. Finally, §7 discusses a possible objection.

1. Introduction

This paper deals with the metaphysics of the notion of quantity in the philosophy of Leibniz, and its aim is to defend the following bi-conditional:

> **(T)** for any object x, x has a certain quantity if and only if x has a limit or a bound.

The notion of quantity at stake here is a mathematical notion, while limit (or bound)[1] is a metaphysical concept. From the 1690s onwards, Leibniz usually defines *Mathesis Universalis* as the theory of quantity: "*Mathesis Universalis* is the science of quantity universally considered, i.e. of the way of estimating, or of designating the limits in the interior of which any thing falls" (GM VII 53)[2]. This theory develops the basic mathematical concepts that lie at the ground of different fields, from number theory (arithmetic is the calculus of determine quantities (or magnitudes), while algebra is the calculus of non-determined quantities), through the *analysis situs,* to his theory of

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estimation of forces in the *Dynamics*. It is therefore hard to overstate the importance of such a notion.

In the present paper, I will not focus on the mathematical theory of quantity, but rather deal with the metaphysical interpretation of such a notion. While this topic has yet to be fully explored, it is one that is fundamental to fully appreciating Leibniz's conception of his General Mathematics. A clarification of the notion of quantity will also illuminate Leibniz's own conception of mathematics.

In §2, I shall start by introducing the notion of quantity with regard to the concepts of part and whole (mereology) and the idea that computing a quantity always requires comparison with another quantity assumed as a fixed measure. I shall argue that these are the two souls of Leibniz's notion of quantity, and that we cannot understand the latter if we do not take both into account. §3 focuses on some passages in which Leibniz explicitly linked the mathematical notion of quantity to the notion of metaphysical limit. We shall see that the link is explicit, and it is already hinted at in the way in which Leibniz defines the notions in play (for example, the same notion of part). This shows that the metaphysics of quantity is not something imposed from an external point of view, but something that can already be found within the mathematical definitions. Moreover, this paragraph justifies the left-to-right direction of the bi-conditional (T). The right-to-left direction is justified in §4, where I present an argument to support it. §5 sketches Leibniz's view of the notion of metaphysical limit in relation to God; this sketch provides further insights on the mathematical notion of quantity (§6). Clearly, the possibility of exploiting Leibniz's conception of the relation between God and the world to clarify the notion of quantity is grounded on my defence of thesis (T). Finally, §7 deals with one objection to the correctness of (T), while §8 concludes.

2. Quantity, Mereology, and Relation

The link between quantity and mereology is a very strong one in Leibniz's philosophy. For example, in different places, he claims that "we can define the quantity of a thing as the property of the whole insofar as it has all its parts" (A VI 4a, 417 circa 1680-1684, GM VII 30 circa 1695). Or, again around 1700, he claims that "it is judged that only those things that are wholes have quantity, in which there are many [parts] homogeneous to the whole" (LH XXXV 1, 9, fol. 1-4. circa 1700?). The claim is thus that the statement "x has a certain quantity" is equivalent to the statement "x is a whole with parts". I shall call this equivalence the Part-Quantity Principle:

Part-Quantity: x has a certain quantity if and only if x is a whole with parts.

The justification of the direction from right to left simply relies on the definition of the part-whole relation. The standard definition of parthood that Leibniz adopted from the 1680s on[3] claims that x is a part of y if and only if y contains x and x is homogeneous with y:

Parthood (1): $P_1xy \equiv Cyx \wedge Hxy$

Where P_1xy must be read as "x is part of y" (the subscript 1 is needed to distinguish this definition from a different one that I present in the next paragraph), Cyx must be read as "y contains x" and Hxy must be read as "x and y are homogeneous". The word "containment" translates the Latin verb "*inesse*" (to be in). Therefore, a necessary condition to be part of a whole is to be in the whole. But this is not enough: the part must be homogeneous with the whole. Homogeneity is here the key notion. Homogeneous things are things that can be compared from a quantitative point of view. In *De Magnitude et Mensura* Leibniz writes that "those things are homogeneous whose magnitudes can be expressed by numbers by assuming the same measure for all of them as unit" (GM VII 35). In *De Quantitate*, we read that "there is proportion only between homogeneous things, and this is clear by definition. If two things are such that one is bigger, smaller or equal to the other, then they are homogeneous" (GM VII 34). Therefore, two things are homogeneous if and only if they are comparable. Since, by definition, the part is homogeneous with the whole, then the part is always comparable to the whole. In the present case of parts and whole, Leibniz endorses the traditional Euclidean common notion according to which the (proper) part is always smaller than the whole. Let us call this principle the Part-Whole Axiom:

Part-Whole Axiom*:* The whole is always bigger than each of its parts.

The Part-Whole Axiom is a statement about the quantities of the whole and the parts: the quantity of the part is always smaller than the quantity of the whole. Clearly this implies that both the part and the whole have a certain quantity. In this way, Leibniz can justify the claim that the statement "x is a whole with parts" implies the statement "x has a certain quantity".

The other direction (from left to right) of Part-Quantity is presented in the quotation above as the definition of quantity. This direction of Part-Quantity is fundamental, since it justifies Leibniz's procedure to compute the quantity of a certain object: "magnitude is that which is expressed in a thing by the numbers of parts congruent with a given thing, which is called measure" (GM VII 35). If an object with quantity were not a whole with parts, then this procedural definition of quantity could not be applied, and we could not compute the quantity (or magnitude) of that object.

Leibniz's procedural definition of quantity tells us that to compute a quantity we must fix an object – that he calls measure – and we must check how many times this object is part of the object whose quantity we want to determine. This shows that the determination of the quantity of an object is always relative to (the quantity of) a different object that we assume as a measure. If we change the measure, the number that expresses the magnitude of the object will change. This relational aspect of quantity is emphasised in the way Leibniz characterises this notion in *Initia rerum mathematicarum metaphysica*, where he claims that "quantity or magnitude is that in things which can be known only through their simultaneous compresence – or by their simultaneous perception" (GM VII 18). We cannot know what quantity is by considering only one object individually; rather, we need a comparison between (at least two) compresent things. Computing a certain quantity always requires comparison with a certain fixed measure.

One should notice that the relational aspect is not in contradiction with the mereological aspect, since mereology tells us how the comparison between different things – an object to be measured and a measure – works. In other words, it is the mereological structure of objects that allows for their quantities to be compared, and so computed.

3. From Quantity to Limit

In a number of writings on *Mathesis Universalis* composed around the end of the seventeenth century and the beginning of the eighteenth century, Leibniz presents a slightly modified definition of part. Part is no longer what is in (*inesse*) a whole and is homogeneous with it; rather, part is what is in the interior (*intra*) of the whole and is homogeneous with it: "part is what is inside a thing (*intra rem*), and at the same time it is homogeneous with it. The thing itself is called whole. To be in something, and to be in the interior of something are different things (*Aliud est esse in re* […] *aliud intra re* […])" (LH XXXV 1, fol. 8 circa 1700, *Mathesis Universalis*, Leibniz 2018, p. 165).

This new definition of the part-whole relation exploits the *intra-esse* relation instead of the *inesse:*

Parthood (2): $P_2xy \equiv Int(x,y) \wedge Hxy$

Where $Int(x,y)$ means that x is in the interior of y.[4] The difference between the *inesse* and the *intra* relation is that the former is reflexive (everything is contained in itself), while the latter is irreflexive (nothing is in the interior of itself). A straightforward consequence is that Parthood (1) allows for an object to be part of itself, while Parthood 2 strictly forbids this from happening: for the latter definition, a part is always different from the whole.[5] In contemporary terms, Parthood (1) corresponds to the notion of improper parthood, while Parthood (2) corresponds to proper parthood. Leibniz then distinguishes two ways in which something can be in the interior of something else. Something can be inside something else as a part, like a triangle inscribed in a circumference, or as a limit, like a line in a figure:

But this latter relation [*intra* relation] is also double: either something is only a limit and it is in this way that the line is in the interior of a figure, or something is a part, as a triangle is in the interior of a circumscribed circle. (LH XXXV 1, fol. 8 circa 1700, *Mathesis Universalis*, Leibniz 2018, p. 165)

From a logical point of view, the two definitions of parts do not differ substantially[6]: in fact, we can define Parthood (1) via Parthood (2), and vice versa:

$$P_1xy \equiv P_2xy \vee x = y$$

$$P_2xy \equiv P_1xy \wedge \neg(x = y)$$

However, the new definition marks an important metaphysical aspect[7]: the stress is now on the fact that whatever has parts also has a boundary. This clearly establishes a connection between the concept of part and the concept of boundary or limit. This link is especially evident in the following passage:

Generally considered, quantity is known when the limits of the thing (*rei*) are determined, in such a way that it is established what is in the interior (*intra*) of it and what is in the exterior (*extrave*) of it. From here we understand that in the same way in which the first philosophy is common to God and the creatures, in this way our *Protomathesis* is a certain common science of the creatures, which are distinguished from God in so far as they are circumscribed by limits. I call "part" what is in the interior of a thing, if it is understood that once subtracted (*ablatum*), it subsists less than the same thing, and the thing "whole". And so the limits, like in the body the point, the line, the surface are not parts […]. (LH XXXV 1, 9, fol. 1-4 circa 1700?, *Scientia Mathematica Generalis*; Leibniz 2018, p. 191)

A part is in the interior of a whole; in turn, the notion of interior requires the notion of exterior, and therefore the notion of limit, bound. This gives us the following conditional statement:

(C1) If x is a whole with parts, then x has a bound, a limit.

Given the equivalence between part and quantity established in Part-Quantity, from C1 we can derive C2:

(C2) If x has a certain quantity, then x has a bound, a limit.

Notice that the notion of limit is clearly a metaphysical notion since, in the passage above, it is evoked to distinguish God from the creatures. Moreover, Leibniz even claims that his "*protomathesis*" (which is how he refers to *mathesis universalis* in this specific paper) is "a common science of the creatures" stressing its metaphysical relevance. But what about the converse of C2? If something is bounded, must it have a quantity? In the next section, I shall present an argument for this conclusion.

4. *From Limit to Quantity*

The above considerations support the view that if x has a certain quantity, then x has a limit. This claim may also be defended by noticing that Leibniz's argument against the infinite number shows that no infinite number exists, since this would imply the existence of an infinite whole, i.e. an infinite quantity.[8] That argument is (also) an argument against the possibility of an infinite quantity. But if there are, properly speaking, only finite quantities, then quantity implies the presence of a limit, a bound. What has no limits at all, i.e. God, has no quantity insofar as it is the maximum of reality. What about the other direction of the implication? Does the property of being bounded or limited imply having a quantity?

Suppose that x has a limit, a bound. This amounts to claiming that x is something created, which belongs to the created world. As we saw above, the boundary is what distinguishes the interior from the exterior and in this way guarantees the determination of a created thing. What is in the interior of an object, namely its ingredients, can be either a part or a limit. A limited thing cannot consist solely of a limit, but the limit is always the limit of something else. Therefore, given a limited thing x, we must be able to find a proper part of x. But at this point Part-Quantity implies that x has a certain quantity. The conclusion is that if something has a bound, then it has a quantity: limit implies quantity. We may schematise the argument as follows:

1. x has a bound (supposition)
2. Having a bound implies having an interior which is separated from an exterior (definition of limit as expressed in the quotation above)
3. What is in the interior is either a limit or a part (premise)
4. All bounded things have parts (since a bound is the limit of something which is not a limit)
5. x has parts (follows from 1, 2, 3, 4)
6. x has a quantity (follows from 5 and Part-Quantity)

The argument has three premises: statements 2, 3, and 4. Statements 2 and 3 are clearly stated in the quotations given in the previous section; statement 4 is not explicitly stated, but it follows from the way in which Leibniz characterises a limit at the end of that quotation. This characterisation is in the negative: a part is what, once subtracted from the whole, leaves "less than the whole", and so everything that, once subtracted, does not leave "less than the whole" is a limit. The idea is the following: if from a circle I subtract a slice with a certain area, I obtain a remainder which is a geometrical figure whose area is strictly smaller than the area of the circle; but if I subtract only the circumference of the circle, the remainder will be a circle with the same area of the original one.[9] From this picture we have two immediate consequences: first, the notion of limit is a relative notion, in the sense that a two-dimensional plane can be the limit of a three-dimensional object, but it can also be approached as a two-dimensional object which has its own parts and its own limits (these limits will be one-dimensional lines); second, no object can be constituted only by its limits. In fact, if an object were constituted only by its limits, the subtraction of all limits should leave us with nothing (by subtracting all limits, we will subtract the entire object, since there is nothing in the object which is not a limit), but this contradicts the characterisation according to which the subtraction of the limits does not give anything "less than the whole". Therefore, the same characterisation of limit implies that something with a limit will also have constituents (or ingredients) which are not limits. In virtue of statement 3, these will be parts. This completes the justification of statement 4.

The conclusion is therefore the following:

(C3) if x is limited, then x has a certain quantity.

Composing together C2 and C3, we obtain the equivalence between "having a quantity" and "having a limit" or "being determined or bounded":

(T) x has a certain quantity if and only if x has a limit or a bound.

5. *The Metaphysics of Quantity*

Since being limited, or bounded, is what distinguishes the creatures from the creator, the consequence of (T) is that quantity only applies to the created world, and not to God. By applying the Part-Quantity principle, this means that God is simple, i.e. there are no parts in it. As we argued above, limit is always a relative notion in the *strict sense* that a limit is always a limit of something else. For Leibniz, a limit is always a limit of a pure attribute (or quality) of God. The idea that I wish to defend now is that quantity originates as soon as God limits his pure positive qualities[10] (in which his attributes consist) to create the world. Quantity is therefore always a quantity of some quality:

> ABSOLUTE is that the concept of which is unlimited or outside of which nothing can be assumed in the same kind, or the concept of which is capable of quantity, and yet does not involve limits. Hence it is possible to conceive absolute Extension, but not an absolute circle. It seems that, in this sense, absolute and maximum are the same. God is the absolute Being [*Ens absolutum*] nor indeed is there any reality or perfection which is not in

God. To put it best, we say that the Absolute is purely positive in its kind. (*Definitiones notionum ex Wilkinsio*, circa 1685–1686, A VI iv 36, quoted from Antognazza 2022).

That the absolute is "capable of quantity" does not mean that there is quantity in the absolute, but rather that God's attributes, which are maximal in the sense of being the greatest of their kind, in the act of creation are bounded and give rise to the creatures which possess only a finite quantity of those attributes. In this way, divine attributes such as extension, duration, goodness, etc. have no limit in God, but they are capable of quantity, namely anything that is an instance of them (God excluded) is indeed limited or bounded, and so will have a certain quantity.

We have here a top-down[11] metaphysical picture, where God is the absolute, pure positive being, with no limitations whatsoever, and the creatures are interpreted as originating from the limitation (or negation) of the positive attributes of God.[12] Limitation is not a positive attribute with an ontological status on its own, but it always presupposes what it is a limitation of. For example, considering the divine attributes of Goodness, Antognazza (2014, p. 180) stresses that "creaturely limitation as the most general case of metaphysical evil is not some kind of reality with a positive ontological status that adversely affects the goodness of creatures. It is merely a way to describe the fact that the goodness of such creatures is limited". In this picture, a limit is something added to a pure positive quality:[13]

Furthermore, of those predicates of things which our mind understands, some are absolute, some involve limitation. Absolute ones consist in a purely positive reality and therefore indicate perfection, and are the only ones counted among the attributes of the supreme substance; limited ones are those to which, in addition to the nature of the absolute, certain circumscribing limits are added, excluding things beyond. (LH 35, 7, 10, Bl. 5r-8v, transcription and translation from Arthur and Ottaviani, *forthcoming*).

Notice that we find again the idea that a limit circumscribes an interior part from an exterior part. A few lines later, dealing with the notion of extension,[14] this idea is strengthened by claiming that a limit in the absolute extension is nothing other than "the negation of further extension":

Thus extension, or, if you prefer, continuous diffusion, is something absolute, which can be understood in the whole and in the part, in the great and in the small; but for a figure, such as a circle or a square, besides extension itself per se, and to that extent taken absolutely, a limit or bound is required, and this is the negation of further extension. (Quoted from Arthur 2021, p. 196)

I take this latter claim to mean that the negation of further extension is simply the circumscribing of a certain area with regard to an exterior area. The notion of limit is thus a relative notion: while circumscribing an interior area, it always implies a reference to an exterior thing. Spinoza's idea of determination as negation (*omnis determinatio est negatio*) expresses a similar thought.

This general metaphysical picture can shed light on the notion of quantity. Recall that our main thesis (T) is an equivalence statement, not an identity. Quantity is a mathematical concept, while limit (in this specific context) is a metaphysical concept. However, what quantity does is to express metaphysical limitation in the mathematical realm. As such, our claim is not that quantity is limit, but that as soon as God, in the creation of the world, limits its purely positive qualities, the creatures originate, and with them the notion of quantity. The creatures are finite instantiations of God's pure positivity, which means that they possess only a (finite) quantity of God's absolute qualities. This clearly implies that there is no quantity in the absolute.

Quantity thus presupposes limitation. Just as a limit is not something with an autonomous ontological status, but expresses the fact that a determined thing participates in the qualitative attributes of God in a non-absolute way, so quantity expresses the fact that a determined thing possesses a certain quality in a non-absolute way. To say that a line is two metres long means to say that it participates of God's attribute of the *immensum* in a limited way.

6. From Metaphysics to the Mathematical Notion of Quantity

Leibniz never defines the notion of metaphysical limit. This is not surprising, since the act of limitation of God's attributes is the same act of creation: therefore, a definition of limit would require taking a direct look at the same act of creation. In this context, with the term "definition" I understand the attempt to explain away a concept through some more basic ones. At most, Leibniz claims that limit is a kind of negation "of further qualities", but what is negation if not a limitation or privation[15] of a positive quality? Something similar happens with quantity, of which Leibniz never gave a proper definition.[16] A case in point is the characterisation that we find in *Initia rerum mathematicarum metaphysica* that it is worth quoting again:

> Quantity or magnitude is that in things which can be known only through their simultaneous compresence – or by their simultaneous perception. (GM VII 18/L 667).

I will use the metaphysical picture sketched in the previous section to explain this characterisation of quantity. In particular, I think that this metaphysical setting can shed light on why such a characterisation does not say what quantity is (it does not explain quantity via more basic terms), but only how we come to know about it. Moreover, it can settle the question whether quantity is a relative notion or not.

According to the metaphysical setting sketched above, quantity is something an object possesses simply because it is bounded. But being bounded is just for the object to exist. Therefore, an object has a quantity simply because it exists. In the same way in which a limit has no positive ontological status but merely indicates a certain restriction (or limitation) of a certain quality, so quantity has no proper ontological status but merely indicates that a pure quality is possessed by an object in a limited, non-absolute way. In this sense, quantity may be interpreted as a relative notion: quantity is always quantity of a certain quality. However, it is not relative in a stronger sense: it is not the case that something has quantity only insofar as it is compared with other objects. Since having quantity just is for an object to be determined, an object has a quantitative aspect independently from any comparison with other objects. Since Leibniz is committed to the idea that the universe is made up of different substances, which are limited (and so finite) instantiations of the pure positivity of God, he is committed to the idea that they have quantity independently of each other.

But doesn't this latter claim make quantity an intrinsic property of an object, contrary to what Leibniz usually says? In fact, he sometimes refers to quantity and number as extrinsic relations, "as mere results, which do not constitute any intrinsic denomination per se, and so they are merely relations which demand a foundation from the category of quality" (quoted from Arthur 2021, p. 190). On one side, if quantity just indicates a limitation of a certain quality, then quantity is grounded on quality, in the sense that quality is conceptually and ontologically prior to it. On the other side, if quantity is an extrinsic relation, then it depends on the relations between different objects. However, here, I think, we should distinguish two cases. The first case is the metaphysical case where we compare a bounded object with the absolute. From the point of view of the absolute, what is real in any object are the qualities it possesses. Since limit and quantity are not positive qualities with their own ontological status, from the point of view of God, they are not intrinsic properties of objects. From the point of view of God, only qualities exist. Moreover, as stressed above, the notion of limit is a relative notion since the delimitation of a certain (interior) area requires to distinguish it from an exterior area. In this sense, quantity too is relative: since x has quantity when x is bounded, and x is bounded implies there is something exterior to it, we cannot avoid this reference to things external to x.

The second case, however, is different. This is the mathematical case in which we compare two bounded (finite) objects in order to give an estimation of the proportion between them. Let us consider a bounded line. That a line A has a length of five metres is clearly an extrinsic property: as explained above, by changing the measure (for instance, from metres to feet), the measure of the length will change (in this case the length will be approximately of 16.4 feet). However, that an object has a quantitative aspect (namely, that it expresses a quality in a limited way) is not something relative to another object, i.e. it is not something that depends on the comparison with other objects. That line A has length does not depend on other lines or on the chosen measure. Line A has length simply because it exists. Therefore, when we focus on the comparison between bounded objects (which is what happens in mathematics or physics), quantity should be seen as a primitive aspect of objects, a sort of intrinsic property that an object has insofar as it exists.

The fact that, at least in the mathematical realm, we should resist a purely relational interpretation of quantity can be appreciated by a number of texts where Leibniz makes clear in which sense quantity is not a relation. For example, in *De Abstracto et Concreto* he claims that:

> Certainly every abstract is either a ground for relation (*fundamentum relationis)* or a relation in itself, namely it is absolute or relative (*respectivum*). That which is absolute is permanent or successive. Permanent is quality or quantity (*qualitas vel quantitas*). Quality is what allows the distinction of things just in virtue of the concept. Quantity is what, in things that have the same quality, i.e. in similar things, can be discerned by the simultaneous experience of two things in the sense of the subject. (A VI 4a 993, my translation)

The passage classifies quantity as a foundation or ground for relation, not as a relation in itself. More specifically, quantity is said to be an absolute, permanent ground for relation. The sense in which quantity is a ground for relation can be clarified by another passage, wherein Leibniz states that "therefore quantity is not a definite number, but the material of number [*materiale numeri*] or an indefinite number which can be determined once we assume a certain measure" (GM VII 31). Quantity as an indefinite number or as the material basis for number is an aspect of objects which is *prior to* the choice of a measure: such a choice and the estimation procedure makes this indefinite number a definite one. As such, quantity is that aspect of objects that makes possible the estimation process.

The distinction between the point of view of God and that of the creatures should also be used in dealing with "Leibniz's shift argument". Leibniz suggests that "if all the things in the world affecting us were diminished in the same proportion, it is evident that not a single person could notice the change" (GM I 180, Arthur 2021, p. 189). The argument works as a *reductio*: we suppose that we can enlarge and diminish the quantities of the things in the universe by maintaining their mutual proportions; we then notice that nobody could spot the differences between the universe before and after the enlargement/diminution, i.e. we would have no means of telling them apart, and so we conclude that this scenario is indeed impossible. But can we conclude that this scenario is metaphysically impossible from the epistemological fact that nobody could spot the difference? This conclusion is drawn by De Risi (2007, p. 358), who stresses that this scenario is indeed impossible according to Leibniz's own metaphysical principles. On one side, I agree with him: from the point of view of God, quantity and limitations are not positive ontological realities, and so God could not discern two universes that differ only in their quantitative aspect, while maintaining all the proportions between the objects. From a qualitative point of view, they would be exactly the same, which means that they would be the same *tout court*. Clearly, the fact that God cannot tell them part is enough to conclude that the proposed situation is an impossibility. And this confirms what we argued before, namely that from the point of view of the absolute, quantity and limitation are relative notions. On the other side, however, De Risi's main reason to claim that this scenario is impossible is that "quantity as such only belongs to the subjective side of perceptual confusion. Thus, a phenomenal world uniformly enlarged would still be exactly the same world for each finite monad. But then, it would be the same world *tout court*, for quantity as a perceptual modality only belongs to a finite representation" (De Risi 2007, pp. 358–59). Here I disagree with De Risi's analysis. From the fact that a finite monad cannot tell the two scenarios apart, De Risi infers that there is no difference at all. This is possible because he interprets quantity as something belonging to the perceptions of finite monads, namely quantity would require the perception of a finite monad: no finite perception, no quantity. A few pages later, he adds: "quantity only belongs to the subjective side of the confused apprehension of quality" (De Risi 2007, p. 402). In this view, finite perception is *prior to* quantity. We explain the latter with the former, not vice versa. However, our thesis (T) gives us a different result: quantity does not depend on the perceptions of finite monads, and so it does not belong to the subjective side of qualities' apprehension. Rather, quantity is as objective as the existence of a plurality of finite substances is. A thing has quantity simply because it exists. From the point of view of the finite substances, quantity is a primitive aspect of the created world, and not a product of those substances' perception. For sure, their perception is confused because it has a quantitative dimension, but this just means that their perception is confused because the substance that perceives is finite. So the relation is reversed: a perception is confused in virtue of its quantitative dimension; we do not explain quantity via the notion of finite perception, but rather we explain finite perception via the notion of quantity. If quantity does not depend on the subjective perception of monads, then the fact that finite monads cannot distinguish the two scenarios imagined in the shift argument is not enough to conclude that the two scenarios are indeed impossible. From the point of view of the finite substances, the diminution in quantity is possible, but not knowable. In this way, the argument shows that what we (as finite monads) know of quantities are only their mutual relationships, and that we do not have direct, i.e. non-relational acquaintance with them.

Connected with this conception of quantity, there is also Leibniz's famous claim according to which God contains all the things eminently, and not formally (Antognazza 2022). This simply indicates that God contains all the positive being of the creatures, and not the creatures with their limitations as they appear to other finite creatures. This does not imply that there is something, namely the limit, that is not contained in God, because – as we stressed above – the limit has no positive ontological status. From the point of view of God, there is therefore no limitation, and so no quantity. God cannot conceive of limitation and quantity. The straightforward consequence is that, strictly speaking, quantity cannot be conceptualised and so can never be captured by means of intellectual concepts. As stressed by De Risi (2007, p. 356), quantity is a "purely sensible notion". The purely sensible element is what prevents us from giving a proper definition of quantity:

> Thus it is impossible for us to know what a foot or a yard is unless we actually have something to serve as a measure which can be applied to successive objects after each other. A foot or a yard can therefore not be explained adequately by a definition; that is, by one which does not include something similar to the thing defined. For though we may say that a foot is twelve inches, the same question arises concerning the inch, and we gain no greater insight, for we cannot say whether the notion of the inch or of the foot is prior in nature, since it is entirely arbitrary which we wish to assume as basic. (GM VII 18/L 669).

However, when we compute a quantity with regard to a fixed measure, the results of the computation can be conceptualised, and indeed *mathesis universalis*, as a theory of quantity, is conceptualisable. But what we are conceptualising are the relations between different quantities, and this relational aspect of quantity is the only thing we can fully grasp of it. In fact, what we normally do is to introduce symbols – like words and digits – to keep track of these relationships between different quantities, as when we say that a line is two metres long, and such symbols are useful in helping our memory to retain such information. In Leibniz's shift argument, this is explained by pointing out that nobody can notice the shift in quantities if the proportions do not change, and this is because what we truly know of quantities is their mutual proportions. However, there is more to quantities than their relations to other quantities: as the passage just quoted makes evident, Leibniz is explicit in claiming that we cannot retain in memory exactly how long one metre, or one foot, is. Since there is a purely sensible aspect that cannot be retained in memory, the only access to it is through a simultaneous (co)perception of different objects. This purely sensible aspect is what, from the point of view of finite monads, is *prior to* any comparison with other quantities.

This explains well Leibniz's characterisation of quantity of *Initia rerum mathematicarum metaphysica*, as what we can know in things when they are simultaneously perceived. First, rather than defining it, Leibniz here is explaining how we know quantity, and this is because there is an aspect of quantity that cannot be conceptualised, and thus cannot be defined. Second, the need of a simultaneous co-perception is due to the fact that this purely sensible notion cannot be retained in memory, and so we cannot properly compare one object with another that we perceived in an earlier instant, since this would require the retention of his quantitative aspect. Third, our main thesis (T) suggests a direct explanation for this situation: quantity expresses a limitation (of a certain quality), and limitation is what pertains to sensible creatures. Quantity and limit are paradigmatic cases of confused notions, not in the sense that they arise from confused perceptions, but in the more radical sense that they are the basis for any confused perception. It is because creatures are limited that they perceive the universe in a confused way. Ultimately, Leibniz's failure to define quantity is not due to some mathematical difficulties, but rather depends on the metaphysical conception of such a notion which is strictly linked with the notion of limit. The difficulty he was facing is essentially a metaphysical one: on one side, the notion of limit is needed to distinguish God from the creatures; on the other side, we must avoid putting the limit in the absolute. The solution is to claim that limit (and quantity) do not have any positive ontological status, and so from the point of view of God, there are only qualities; however, from the point of view of the creatures, they are something present and not eliminable. The fact that quantity, from the point of view of the finite monads, is a primitive aspect of finite objects which cannot be directly conceptualised (and so what we can know about quantity is only relative to a comparison between different quantities) precisely indicates this peculiar status. According to this interpretation, quantity is nothing other than a mathematical correlate of this metaphysical notion of limit.

7. *Bounded Infinities and the Syncategorematic Infinite*

There is an important objection to the view just stated, which will require us to dig deeper into a number of Leibniz's metaphysical views. The objection runs as follows: Leibniz admitted both the notion of a bounded infinite, and the presence of the actual infinite in nature. These two cases show that "being limited" is compatible with "being infinite" but, properly speaking, there are only finite quantities. If being limited implies having a (finite) quantity, then Leibniz could admit neither the notion of bounded infinities nor the actual infinite in the created world.

In *De Quadratura*, Leibniz distinguishes between unbounded and bounded infinite lines (*infinita terminata*). An unbounded infinite line is a line which does not have a last point (at least on one side); this line can never be exhausted by the repetition of a finite line. On the contrary, a bounded line is constituted by some finite line, no matter how many times it is repeated. In the case that a finite line is repeated more times than can be expressed by any number, then the resulted bounded line would be infinite (Knobloch 1986, p. 42). Clearly, a bounded infinite line is infinite insofar as its length is infinite, namely it corresponds to an infinite quantity. But as we know, an infinite quantity would contradict the Part-Whole Axiom, and so a bounded infinite line must be a fictional entity: we feign that a bounded infinite line is a proper object, but instead it is only a shortcut that allows us to consider a line bigger than any preassigned line which can be compared with finite lines (that is the difference between unbounded and bounded lines).

If bounded infinites are only fictions, then they fail to show that bounded is compatible with infinite, which would contradict the idea that bounded and quantity are equivalent. Moreover, if bounded infinites were a consistent notion, then the notion of infinite quantity would be consistent, and therefore we would still not have a contradiction with the claim that quantity is equivalent to being limited.

The second part of the objection regards the presence of the actual infinite in the created world. However, again, as Richard Arthur has extensively argued (Arthur 2018a, 2018b), the infinite in question is actual, but syncategorematically understood. This simply means that there are infinitely many things (infinitely many monads, infinitely many parts of a body, infinitely many perceptions of a single monad), but these infinites do not constitute a further object (a collection or a whole) with regard to their constituents. There are more things than can be counted, but none of these things has an infinite quantity. The distinction is between an infinite in magnitude and an infinite in multitude. The former is more than any assignable number of finite quantities, but it is for Leibniz only a fiction; strictly speaking, Leibniz's metaphysics is committed only to the infinite in multitude, as I am going to argue. Let us start our examination with bodies, and then move on to monads.

It is well known that Leibniz considers a body an infinite aggregate of parts, which in turn are smaller bodies. A body is actually divided into infinitely many parts, where each division is the result of the movements that animate the body. These movements are the physical expressions of some substantial forms in the body. Despite being an infinite aggregate, it is false to claim that a body has an infinite number of parts; what we are allowed to say is that there are infinitely many parts, namely the parts of a body exceed any (finite) number. This is the actual infinite syncategorematically understood.

However, one may still wonder whether the claim that a body has infinitely many parts implies the claim that the magnitude of a body is infinite. The magnitude is what we compute when we look at how many times a fixed object called measure (congruent to a part of the object whose magnitude we are looking for) is contained in it. Taking any finite measure, an object would be infinite if the measure is contained infinitely many times in it. Since a body has parts, we can take each of these parts as a measure; in order for its magnitude to be infinite, it is enough that one finite part is contained in it infinitely many times. If this is the case, then it cannot happen that another finite part is contained in it finitely many times.

If a body were divided into infinitely many finite *disjoint* parts, then its magnitude would be infinite. Two parts are disjoint if they have nothing in common. If, for simplicity, we suppose that the division is highly uniform, i.e. each part is congruent with each other, then a single part would be contained infinitely many times in a body. But the division of a body into parts is not of this type. Each body is divided into some finite parts, each of which are further divided into some other finite parts, which will be divided into further parts and so on. The parts are not generally disjoint, since we find parts inside other parts. What we find here is a kind of recursive structure, where at each level of the division we always have finitely many parts. The infinite of the parts depends on the fact that there are infinitely many levels of divisions. But at each level we only have a finite number of parts.[17] This implies that we may consider any part of a body, and it will be contained a finite number of times in the whole body. Consequently, the magnitude of a body is finite, despite it having infinitely many parts.[18]

What about substances? Apart from God, each substance is bounded; however, it is also said to be infinite: for example, each substance has infinitely many perceptions, since it perceives the entire universe (which is an aggregate of infinitely many things). But once again, the infinite here is syncategorematic: the universe is not a whole (or an organism) and therefore to perceive each state of the universe means to perceive infinitely many things, not to perceive an infinite thing.

These considerations do not seem to pose any worries with regard to my main claim in this paper. Everything which is bounded has a (finite) quantity, and everything with a quantity is bounded. The infinite present in the world is an actual syncategorematic infinite, which just requires that there are infinitely many things, but not that there is an infinite object. My conclusion is that this kind of infinite is fully compatible with the admission that quantity can only be finite, and so with the bi-conditional (T).

8. Conclusion

In this paper, I have presented a metaphysical interpretation of the mathematical notion of quantity, which consists in the claim that the statement “x has a certain quantity” is equivalent to the statement “x is limited or bounded”. First, I defended this view by looking at different texts on *Mathesis Universalis* which show that this equivalence already emerges in a mathematical context, namely from the mathematical definition of notions such as quantity, part-whole, the intra-relation, etc. I have individuated some of the key principles of Leibniz’s mathematical theory of quantity (such as Part-Quantity or the Part-Whole Axiom), and used them to argue for my main claim.

The fact that (T) connects the mathematical level with the metaphysical one, and emerges already in the mathematical context, allows us to look at metaphysics to clarify the mathematical notion of quantity. From a metaphysical point of view, quantity just expresses the limitation of a certain quality. The reconstruction of Leibniz's top-down metaphysics (from the absoluteness of God to the limitation of the world) enables us to clarify a number of questions pertaining to the characterisation of quantity, in particular the last characterisation from *Initial rerum mathematicarum metaphysica*. Specifically, I argued that Leibniz's struggle in finding a suitable characterisation of quantity is just a reflection of a metaphysical difficulty that any top-down metaphysics faces, that of explaining how the finite, the limited, and the privative can originate from the absolute perfection and the absolute simplicity of God.

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[1] Here, and in the rest of the paper, I shall use the words "limit" and "bound" as synonyms.

[2] If not otherwise stated, the translations from Latin are mine. Already in this passage, we can appreciate the link between quantity and limit.

[3] Despite a different definition that appears around 1700 (and that is discussed in the next paragraph), Leibniz still presented his standard definition in 1715 in *Initia rerum mathematicarum metaphysica*.

[4] This notion of interior is very close to the notion of interior in contemporary topology. The interior of a set X, denoted as *Int(X)* is defined as the union of all open sets included in X: $Int(X) = \cup_i B_i$

where B_i is an open set such that $B_i \subseteq X$. The idea is that an interior does not contain the limit points, where x is a limit point of X if and only if any open neighbourhood containing x intersects $X \setminus \{x\}$. The union of the interior and the limit points gives us the closure of a set, which is equal to the intersection of all closed sets containing the set. The boundary of a set is usually defined as the intersection of the closure and the closure of the complement. This concept can be used to give a mathematical interpretation of the metaphysical notation of limit or boundary that we are dealing with. Clearly, the notion of metaphysical bound should be kept distinct from the notion of a bounded set, since the latter depends on a given metric.

[5] In this text, Leibniz does not make explicit the formal feature of the *intra* relation. However, since the *inesse* relation is reflexive, antisymmetric, and transitive (see, e.g., Mugnai 2019), and the only difference consists in the irreflexivity of the *intra* relation, it follows that such a relation is irreflexive, asymmetric, and transitive.

[6] The justification of the claim "'x is a whole with parts' implies 'x has quantity'" that we gave in section 2 was based on Parthood (1), but it can be directly stated also with Parthood (2).

[7] This is not the first time that Leibniz linked the notion of quantity with the notion of limit (and the *intra* relation). For example, already in 1679, he had written: "*Quantitas exhibit modum cognoscendi quaenam sint rei partes, quantitas praebet modum inveniedi rei terminos, et quid sit intra aut extra ipsam: non consideratis formis seu qualitativas*" (A VI, 4, 153). Commenting on this passage, Rabouin (in Leibniz 2018, p. 192, footnote 3) claims that in 1679 this was only one of several different characterizations of quantity.

[8] On Leibniz's argument against infinite number see, for example, Van Attend (2010) or Costantini (2020).

[9] The notion of subtraction that Leibniz is here using is not arithmetical subtraction that applies only to magnitudes, but is the notion of Real Subtraction, which he had introduced in the papers on Real Addition. This notion applies to any kind of object. On the Real Addition calculus, see Lenzen (1989) and Mugnai (2019).

[10] The expression "pure positivity" indicates the absence of any limitation or negation. For a deep analysis of this notion, see Antognazza (2022).

[11] See Antognazza (2015) for a deep analysis of the notion of God as the hypercategorematic infinite (namely the infinite which is beyond any category).

[12] On this point, see Antognazza (2014, p. 180), and Arthur (2021, §2.5).

[13] The idea of a limit or a bound as an addition is also expressed in the following passage: "In the continuum, the whole is prior to its parts: the absolute is prior to the limited: and so the unbounded is prior to that having bound, since a bound is a kind of addition" (A VI 3 502).

[14] The notion of extension has its divine basis on the divine attribute of immensity, which, insofar as it is a divine attribute, is infinite, indivisible, and unbounded: on this point, see Arthur (2021, pp. 147–50).

[15] Another important connection is that between limitation, privation, and metaphysical evil. I will not develop this point here. On this issue, see Antognazza (2014, 2015, 2022).

[16] This latter claim may simply sound wrong, since Leibniz has defined quantity in different ways throughout his life. However, either these definitions limit themselves to giving a procedure to compute a certain quantity once a measure has been fixed, or, like the one with which we began this paper, these definitions just limit themselves to saying that quantity is a property of a whole, but fail to explain which property it is. None of Leibniz's characterisations of quantity explains the notion away by means of some more primitive concepts.

[17] Arthur (2018b, chapter 2) extensively argues for the same thesis.

[18] There would be a further problem for the idea that the magnitude of a body is infinite. The magnitude is expressed by the number of parts of an object congruent with a measure. Therefore, to express an infinite magnitude, we need an infinite number.