Expectations, cores, and strategy-proofness under externalities*

María Haydée Fonseca-Mairena† Matteo Triossi‡

Abstract

We study the connection between cores and strategy-proofness in environments with externalities. With this objective in mind, we present a new concept of the core that relies on agents' expectations about their peers' reactions to group deviations. It encompasses several core consistent solutions previously proposed in the literature for environments with externalities. It allows us to prove that essentially single-valued cores are necessary and sufficient for the existence of strategy-proof, efficient, and individually rational mechanisms.

Keywords: Allocation problems, Core, Expectations, Externalities, Strategy-proofness.

Economic Literature Classification Numbers: C71, C72, C78, D62, D78.

*This paper is a modified version of the third chapter of the Ph.D. dissertation of the first author at the University of Chile. We thank Nadia Buriani, Damián Gibaja, Lars Ehlers, Nicolás Figueroa, Daniel Hojman, Rahmi Ilkilic, Maciej Kotowski, Alfonso Montes, Emma Moreno, Antonio Romero-Medina, Juan Pablo Torres-Martínez, and William Thomson for useful comments. We also thank participants of the Latin American Workshop in Economic Theory, the Lisbon Meeting in Game Theory and Applications, the Meeting of the Society for Social Choice and Welfare, the Stony Brook International Conference on Game Theory, and the GRASS Workshop. We acknowledge financial support from the Institute for Research in Market Imperfections and Public Policy, ICM IS130002, Ministerio de Economía, Fomento y Turismo. We also thank the University of Chile Department of Economics for the hospitality while completing the paper. Triossi acknowledges financial support from Ca' Foscari University of Venice under the project MAN.INS_TRIOSSI and the Overseas Program. The authors declare they do not have any conflict of interest.

†Corresponding author. Department of Economics and Management, Universidad Católica del Maule. San Miguel 3605, Talca, Chile. E-mail: mfonseca@ucm.cl.

‡Department of Management, Ca' Foscari University of Venice. Fondamenta San Giobbe, Cannaregio 873, 30121 Venice, Italy. E-mail: matteo.triossi@unive.it.

1 Introduction

Externalities play a crucial role in real-world allocation problems. Notable instances include labor markets, where workers are concerned about their colleagues, and school choice problems, where families consider the impact of their children's classmates. In team sports competitions, the composition of other teams is a crucial concern for each player, as it directly influences their own team's outcomes and rankings. Similarly, in cartel formation problems, the profits of cartel members are affected by the activities of other cartels and independent firms. Likewise, in international trade, the benefits of fiscal coordination depend on the behavior of countries outside the alliance.

The contribution of the paper is twofold. Firstly, we introduce a general model of expectations to address externalities. Secondly, we employ this model to analyze the connection between cores and strategy-proofness in environments with externalities.

The seminal contribution by Sönmez (1999) proves that the existence of a strategy-proof, efficient, and individually rational mechanism constrains the core to be essentially single-valued. That is, all allocations in the core are indifferent to all agents. On the other hand, an essentially single-valued core guarantees that all its selections are strategy-proof, efficient, and individually rational mechanisms. While these results also apply in environments with externalities, they have no grip if the core is empty, which is often the case when externalities are relevant (Chander and Tulkens (1997), Roth and Sotomayor (1992), Sasaki and Toda (1996), Mumcu and Saglam (2007), Ehlers (2018)).

The literature has dealt with the existence problem by relaxing the core's requirements.[1] Ehlers (2018) extends the analysis of Sönmez (1999) by considering the individually rational core, introduced in Hart and Kurz (1983) as the γ-core (see also Chander and Tulkens (1997)). It assumes that if a coalition deviates, all other agents will receive their individual endowments back. Hart and Kurz (1983), Sasaki and Toda (1996), and

[1] An alternative approach is to consider restrictions on preferences that guarantee the nonemptyness of the core. See Echenique and Yenmez (2007) and Dutta and Masso (1997) in college admission problems, Alcalde and Revilla (2004) in the formation of research teams, Klaus and Klijn (2005) and Bando (2012) in labor markets, Mumcu and Saglam (2010) in marriage markets, Hong and Park (2022) and Salgado-Torres (2011) in housing markets, Pycia and Yenmez (2023) in matching with contracts. Bando et al. (2016) surveyed the literature about two-sided matching markets with externalities.

Mumcu and Saglam (2010) introduce alternative solutions. These works make ad-hoc assumptions about how the agents outside of a deviation coalition react to the deviation. It is crucial because, under externalities, this reaction will affect the agents' payoff within the deviating coalition. Thus, it will influence the decision to deviate.

To better understand this decision, we explicitly introduce expectations about the behavior of the agents not belonging to a deviating coalition. Consider a situation where the agents receive allocation a. Assume the agents in coalition T redistribute their respective endowments and attempt to implement allocation b. We define expectations as the set of allocations that the agents in T believe can be achieved by trying to implement allocation b.[2] We then define a core based on the agents' expectations. We assume that a coalition of agents deviates from a given allocation if any of the expected outcomes of the deviation makes no coalition member worse off and makes at least one coalition member strictly better off. The model encompasses several core consistent solutions previously introduced to deal with externalities.

Larger expectations, in the set inclusion order, generate larger cores. Thus, the model allows for determining the minimal and maximal cores. The smallest core is the "optimistic core". It is consistent with optimistic expectations: the agents expect that, following the deviation of coalition T, the agents outside of the coalition will follow the proposal of coalition T. It coincides with the usual definition of core employed in Sönmez (1999). Considering expectations such that all coalitions can modify any allocation, the largest core is the "prudent core". It is consistent with prudent expectations: the agents expect that, following the deviation of coalition T, the agents outside the coalition can redistribute the endowments in all conceivable manners.[3] Both the optimistic and the prudent core subsets of the set of efficient and individually rational allocations, which can be modeled as a core as well.

[2] Bloch and van den Nouweland (2014) consider expectations in coalition formation games in partition function form. Differently from us, they consider single-valued expectations only. See also Bloch and van den Nouweland (2020).

[3] In the literature, prudent agents have been also labeled as "pessimists". Prudent expectations are related to the α-effectiveness introduced by Aumann and Peleg (1960). They are also consistent with Sasaki and Toda's (1996) bilateral stability concept for marriage markets. See also Hafalir (2007), Contreras and Torres-Martínez (2021), and Fonseca-Mairena and Triossi (2019, 2022, 2023).

Then, we study the relationship between cores and strategy-proof, individually rational, and efficient mechanisms (SIEM for short) under externalities.

We extend and connect the work of Sönmez (1999) and Ehlers (2018) (see also Takamiya (2003)) and prove that if an $SIEM$ exists, the correspondence of efficient and individually rational allocations is essentially single-valued (Theorem 1). In particular, all cores are essentially single-valued. Differently from Sönmez (1999) and Ehlers (2018), who limit the externalities related to the endowment, our assumptions enable us to deal with strict preferences under externalities.

Furthermore, we prove that if any of the cores is externally stable, non-empty, and essentially single-valued in a given preference domain, any core selection is strategy-proof, efficient, and individually rational (Theorem 2). The result thus generalizes Sönmez (1999) and Ehlers (2018) (see also Demange (1987) and Takamiya (2003)).

We present applications of the results to coalition formation problems and marriage markets. In coalition formation problems, we identify a preference domain with externalities in which an $SIEM$ exists, the domain of block preferences in which agents first care about the bundle goods they receive. In marriage markets, we first prove an impossibility result for agents on one side of the market having common preferences over matchings. In addition, we demonstrate that under block preferences, an $SIEM$ exists if the preferences of the agents on one side of the market are acyclic.

Finally, we discuss the implication of relaxing the individual rationality constraint for our results.

The paper proceeds as follows. Section 2 presents the model. Section 3 studies the relationship between cores and $SIEM$. Section 4 presents applications to coalition formation problems and marriage markets. Section 5 presents and discusses relaxing the individual rationality constraint. Finally, Section 6 concludes. Appendix A includes additional examples of expectations and cores. The other appendices include the proofs omitted from the main text.

2 The Model

There is a finite set of agents, N, $|N| \geq 3$.[4] Agent $i \in N$ owns an individual endowment e_i, which is a set of indivisible goods. An allocation is a correspondence $a : N \rightrightarrows \bigcup_{i\in N} e_i$ such that $\bigcup_{i\in N} a(i) = \bigcup_{i\in N} e_i$ and $|a^{-1}(x)| = 1$ for all $x \in \bigcup_{i\in N} e_i$. Allocation e, defined by $e(i) = e_i$ for all $i \in N$, is called the endowment. Set $\mathcal{A}$ is the set of all allocations, and $\mathcal{A}^f \subseteq \mathcal{A}$ is the set of feasible allocations. We assume $e \in \mathcal{A}^f$.

Every agent $i \in N$ has preferences R_i, a complete and transitive binary relation over $\mathcal{A}^f$. Let $R = (R_i)_{i\in N}$ be a preference profile. For every i, we denote by P_i and I_i, the strict and the indifference relations associated with R_i, respectively. Let $L(a, R_i) = \{b \in \mathcal{A}^f : aR_ib\}$ be the lower contour set of a at R_i, and let $L^*(a, R_i) = \{b \in \mathcal{A}^f : aP_ib\}$ be the strict lower contour set of a at R_i. Let $\mathcal{R}$ be the set of preferences over $\mathcal{A}^f$. Let $\mathcal{D} \subseteq \mathcal{R}^{|N|}$ be the set of admissible preference profiles. Let $\mathcal{P}$ be the set of strict preferences on $\mathcal{A}^f$.

Remark 1. *The model is able to describe several allocation problems, including:*

(i) *Coalition formation. For each $i \in N$, let $e_i = \{\omega_{ij} : j \in N \setminus \{i\}\}$, in which ω_{ij} represents the permission for agent j to join a coalition to which agent i belongs. Let $\mathcal{A}^f \subseteq \mathcal{A}^C = \{a \in \mathcal{A} : \omega_{ij} \in a(j) \Rightarrow \omega_{ji} \in a(i)\,,$ and $\forall i, j, k \in N, \omega_{ij} \in a(j)\,, \omega_{jk} \in a(k) \Rightarrow \omega_{ik} \in a(k)\}$.*

(ii) *Housing market: $|e_i| = 1$ for all $i \in N$ and $\mathcal{A}^f \subseteq \mathcal{A}^H = \{a \in \mathcal{A} : |a(i)| = 1, \forall i \in N\}$.*

(iii) *Roommate problem: $e_i = \{i\}$ for all $i \in N$ and $\mathcal{A}^f \subseteq \mathcal{A}^R = \{a \in \mathcal{A}^H : a(i) = \{j\} \Leftrightarrow a(j) = \{i\}, \forall i, j \in N\}$.*

(iv) *Marriage market: $W \cup M = N$ in which W and M are two disjoint sets, $e_i = \{i\}$ for all $i \in N$, and $\mathcal{A}^f \subseteq \mathcal{A}^{MR} = \{a \in \mathcal{A}^R : \forall m \in M, \forall w \in W, a(w) \in M \cup \{w\}, a(m) \in W \cup \{m\}, a(w) = \{m\} \Leftrightarrow a(m) = \{w\}\}$.*

Allocation a is **individually rational** if aR_ie for all $i \in N$. Let $\mathscr{I}(R)$ denote the set of individually rational allocations. A feasible allocation a is **efficient** under R when there

[4] Denote by $|X|$, the cardinality of set X.

is no $b \in \mathcal{A}^f$ such that bR_ia for all $i \in N$ and bP_ja for some $j \in N$. Let $\mathscr{E}(R)$ denote the set of efficient allocations and let $\mathscr{E}\mathscr{I}(R) = \mathscr{E}(R) \cap \mathscr{I}(R)$ be the set of efficient and individually rational allocations.

Let $\mathcal{D}_i \subseteq \mathcal{R}$ for all $i \in N$. Let $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$. A mechanism $\Gamma : \mathcal{D} \to \mathcal{A}^f$ is **strategy-proof** if being truthful is a weakly dominant strategy for all agents, formally if, for all $i \in N$, $R \in \mathcal{D}$, and for all $R'_i \in \mathcal{D}_i$, $\Gamma(R)\, R_i\, \Gamma(R'_i, R_{-i})$. A mechanism Γ is **weakly coalitionally strategy-proof** if for all $R \in \mathcal{D}$, for all $T \subseteq N$, and for all $R'_T \in \mathcal{D}_T = \prod_{i \in T} \mathcal{D}_i$ there exists $i \in T$ such that $\Gamma(R)\, R_i\, \Gamma(R'_T, R_{-T})$.

A mechanism Γ is individually rational if $\Gamma(R) \in \mathscr{I}(R)$ for all $R \in \mathcal{D}$. A mechanism Γ is efficient if $\Gamma(R) \in \mathscr{E}(R)$ for all $R \in \mathcal{D}$. A strategy-proof, individually rational, and efficient mechanism is called $SIEM$.

A correspondence $\Omega : \mathcal{D} \rightrightarrows \mathcal{A}^f$ is essentially single-valued if, for all preference profiles R, two allocations in $\Omega(R)$ are indifferent for all agents. Formally, Ω is **essentially single-valued** if, for each $R \in \mathcal{D}$, $a, b \in \Omega(R) \Rightarrow aI_ib$ for all $i \in N$.

2.1 Expectations

We define expectations as the set of allocations that the agents in a deviating coalition believe can be achieved by trying to implement allocation b (the "announcement" from now on) from an initial allocation a. Formally, the **expectations** of agent i is a correspondence $\Theta_i : \mathcal{A}^f \times \mathcal{A}^f \times 2^N(i) \times \mathcal{D} \rightrightarrows \mathcal{A}^f$. Here, $2^N(i)$ is the set of coalition to which agent i belongs, $2^N(i) = \{T \subseteq N : i \in T\}$.[5] [6]

An **allocation problem** is a tuple $\mathscr{A} = \left(N, e, \mathcal{A}^f, R, (\Theta_i)_{i \in N}\right)$. It has externalities if there exists $i \in N$, $a, b \in \mathcal{A}^f$ such that $a(i) = b(i)$ and aP_ib.

If coalition T deviates from allocation a and tries to move to allocation b, the agents outside T may react. We assume that the members of T deviate if and only if no agent in

[5]The estimations/conjectural valuations in Sasaki and Toda (1996), the expectation in Bloch and van de Noweland (2014, 2020), and the beliefs in Braitt and Torres-Martínez (2021) and Piazza and Torres-Martínez (2024) are examples of expectations.

[6]Agents can assign probabilities to the allocations that may be reached as a result of the reactions of the agents outside the deviating coalition T. In this case, expectations can be interpreted as (sets of) probability distributions. We do not explicitly consider this environment, whose study is left for future research.

T is made worse off in any of the allocations that they expect can be attained and at least one agent in T is strictly better off in all allocations that she expects can be attained.

Definition 1. *Let $R = (R_i)_{i\in N}$. Coalition T* ***blocks*** *allocation $a \in \mathcal{A}^f$ announcing $b \in \mathcal{A}^f$ if:*

(i) *for all $i \in T$, cR_ia for all $c \in \Theta_i(a, b, T, R)$;*

(ii) *there exists $i \in T$ such that cP_ia for all $c \in \Theta_i(a, b, T, R)$.*

We say that coalition T blocks allocation $a \in \mathcal{A}^f$ if there exists $b \in \mathcal{A}^f$ such that coalition T blocks allocation $a \in \mathcal{A}^f$ announcing b.

The **core** of an allocation problem $\mathscr{A}$ is the set of unblocked and individually rational allocations and is denoted by $\mathscr{C}(\mathscr{A})$. When there are not ambiguities about $\mathscr{A} = \left(N, e, \mathcal{A}^f, R, (\Theta_i)_{i\in N}\right)$, we will simply write $\mathscr{C}(R)$ instead of $\mathscr{C}(\mathscr{A})$. Given, $(N, e, \mathcal{A}^f, (\Theta_i)_{i\in N})$, the core correspondence $\mathscr{C} : \mathcal{D} \rightrightarrows \mathcal{A}^f$ assigns to each preference profile $R \in \mathcal{D}$ the core $\mathscr{C}(R)$. Notice that different expectations may generate the same core correspondence.

The core correspondence is efficient if $\mathscr{C}(R) \subseteq \mathscr{E}(R)$ for all $R \in \mathcal{R}$. [7]

We impose a minimal consistency requirement on expectations that we call admissibility. We assume the agents expect that the deviation of a coalition, T, produces an effect only if the agents of T do not attempt to take (part of) the endowment of agents outside the coalition and that the agreement within coalition T is not modified.

Formally, expectation Θ_i is **admissible** if $\Theta_i(a, b, T, R) \neq \{a\}$ implies:

(i) $\bigcup_{j\in T} b(j) = \bigcup_{j\in T} e_j$;

(ii) for all $c \in \Theta_i(a, b, T, R)$, $c(j) = b(j)$ for all $j \in T$.

Throughout the paper, we consider only admissible expectations generating efficient core correspondences.

The literature has often dealt with blocking concepts in which every coalition can modify the current allocation. Not all admissible expectations have this characteristic. Let $T \subseteq N$, expectation Θ_i is T-effective if $\bigcup_{j\in T} b(j) = \bigcup_{j\in T} e_j$ implies $\Theta_i(a, b, T, R) \neq \emptyset$

[7] If $\Theta_i(a, b, N.R) = \{b\}$ for all $b \in \mathcal{A}^f$ and all $i \in N$, the core is efficient.

and $\Theta_i(a,b,T,R) \subseteq \{c : c(i) = b(i)\,,\ \text{for all } i \in T\}$, for all $a,b \in \mathcal{A}^f$. Expectations are **proper** if they are T-effective for all $T \subseteq N$.

Next, we present admissible expectations that model concepts of cores previously considered in the literature and will be employed in the paper. Let $a,b \in \mathcal{A}^f$, $T \subseteq N$, $i \in T$, and $R \in \mathcal{D}$.

(i) *Efficient expectations*: $\Theta_i^E(a,b,T,R) = \{b\}$ if $T = N$, $\Theta_i^E(a,b,T,R) = \{a\}$, otherwise. Expectations Θ_i^E are N-effective but not T-effective for all $T \subsetneq N$. Then, efficient expectations are not proper. If all agents have efficient expectations, the core is $\mathscr{E}\mathscr{I}(R)$.

(ii) *Prudent expectations*: $\Theta_i^P(a,b,T,R) = \left\{c \in \mathcal{A}^f : c(k) = b(k), \forall k \in T\right\}$ if $\bigcup_{k\in T} b(k) = \bigcup_{k\in T} e_k$, $\Theta_i^P(a,b,T,R) = \{a\}$, otherwise.

Prudent agents belonging to a coalition, say T, do not know how the agents outside T will react to a deviation. Thus, they deviate if and only if all members of T are weakly better off and at least one member of T is strictly better off as a result of the deviation, independently of the behavior of the agents outside T. This behavior is consistent with an extreme form of uncertainty aversion (see Ellsberg (1961) and Gilboa and Schmeidler (1989)). It is also consistent with an extreme form of pessimism, according to which each agent in a deviating coalition, T, believes that the agents outside T act to minimize her welfare.[8] If all agents have prudent expectations, the core is called "prudent" and denoted by $\mathscr{C}^P(R)$. The prudent core coincides with the α-core (see Aumann (1961) and Hart and Kurz (1983)).[9] Prudent expectations are proper.

(iii) *Individually rational expectations*: $\Theta_i^{IR}(a,b,T,R) = \{c^{IR}(b,T)\}$ if $\bigcup_{i\in T} b(i) = \bigcup_{i\in T} e_i$ and $c^{IR}(b,T) \in \mathcal{A}^f$, $\Theta_i^{IR}(a,b,T,R) = \{a\}$, otherwise. In which $c^{IR}(b,T)(k) = b(k)$ for all $k \in T$, $c(b,T)(k) = e(k)$ for all $k \notin T$.

[8] In the coalition formation games analyzed in Bloch and van den Nouweland (2014), prudent expectations are generated by what they call the "pessimistic rule".

[9] Fonseca-Mairena and Triossi (2022) study the implementability of the α-core in Nash equilibrium. Sasaki and Toda (1996) define pairwise stable allocations in matching market with externalities and prudent agents. See also Contreras and Torres-Martínez (2021) and Hong and Park (2022).

If coalition T forms and proposes b, each agent outside T receives her individual endowment. Individually rational expectations are proper if and only if $c^{IR}(a,b) \in \mathcal{A}^f$ for all $a, b \in \mathcal{A}^f$. If all agents have individually rational expectations, the core is called "individually rational" and denoted by $\mathscr{C}^{IR}(R)$. The individually rational core coincides with the IR-core studied by Ehlers (2018) (see also Hart and Kurz (1983)).

(*iv*) *Optimistic expectations*: $\Theta_i^O(a,b,T,R) = \{b\}$ if $\bigcup_{i\in T} b(i) = \bigcup_{i\in T} e_i$, $\Theta_i^O(a,b,T,R) = \{a\}$, otherwise.

Optimistic agents expect that the agents in a deviating coalition, T, can achieve any allocation consistent with redistributing their endowments. If at least one of such allocations makes all members of T weakly better off and a member of T strictly better off, coalition T deviates. Optimistic expectations are proper. This behavior is consistent with an extreme form of optimism according to which each agent in a deviating coalition, T, believes that the agents outside T act to maximize her welfare.[10] An alternative interpretation is that, if the agents have optimistic expectations they believe deviating coalitions can also determine the objects the agents outside the coalition will receive. If all agents have optimist expectations, the core is called "optimistic" and denoted by $\mathscr{C}^O(R)$. The optimistic core coincides with the usual core employed in Sönmez (1999) and with the ω-core studied in Kóczy (2007). Optimistic expectations are proper.

Appendix A discusses other expectations that have been considered in the literature.

3 Results

3.1 The biggest and the smallest core

We start studying the relationship between the cores generated by different expectations.

[10] In the coalition formation games analyzed in Bloch and van den Nouweland (2014), optimistic expectations are generated by what they call the "optimistic rule".

Optimistic expectations always yield the smallest core. Also, if expectations are proper, prudent expectations yield the largest core.

Proposition 1. *Given* $\left(N, e, \mathcal{A}^f, (\Theta_i)_{i \in N}\right)$*, let* $R \in \mathcal{R}$*.*

(i) $\mathscr{C}^O(R) \subseteq \mathscr{C}(R) \subseteq \mathscr{E}\mathscr{I}(R)$*;*

(ii) if $(\Theta_i)_{i \in N}$ *are proper,* $\mathscr{C}(R) \subseteq \mathscr{C}^P(R)$*.*

Item (i) follows from the definition of optimistic expectations. The proof of item (ii) follows from the fact that larger expectations yield larger cores (Lemma 7 in the Appendix) and that $\Theta_i \subseteq \Theta_i^P$ for all $i \in N$ if expectations $(\Theta_i)_{i \in N}$ are proper (Lemma 8 in the Appendix).

The following diagram illustrates the relationships underlined in Proposition 1.

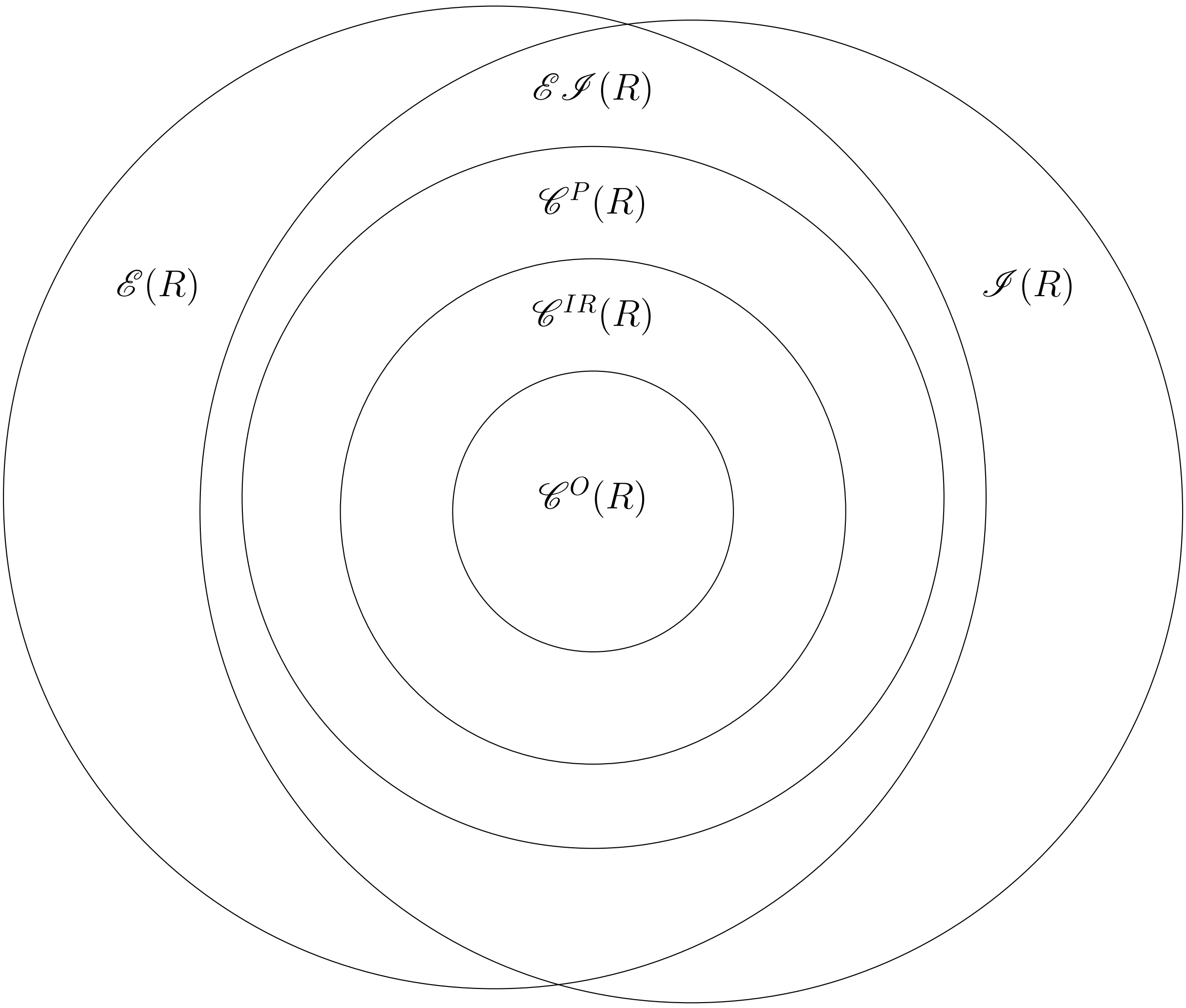

Figure 1.

If the expectations are not proper, the inclusion $\mathscr{C}(R) \subseteq \mathscr{C}^P(R)$ does not necessarily hold.

Example 1. *Let $N = \{1,2,3,4\}$. Let $e = (e_1, e_2, e_3, e_4)$, $a = (e_2, e_1, e_4, e_3)$ and $\mathcal{A}^f = \{a, e\}$. Let the preferences be:*

$$R_1 : aP_1e; \qquad R_2 : aP_2e; \qquad R_3 : eP_3a; \qquad R_4 : aP_4e.$$

We have $\mathscr{C}^P(R) = (R) = \emptyset$ but $\mathscr{C}^{IR}(R) = \{e\}$.

In Example 1, claim (b) of Proposition 7 fails because $\mathcal{A}^f$ does not contain an allocation different from e in which agent 3 keeps her endowment, in particular does not contain $c(a, \{1,2\}) = (e_2, e_1, e_3, e_4)$, which implies $\Theta_i^{IR}(e, a, \{1,2\}, R) = \{e\}$. Here, individually rational expectations are not proper because they are not $\{1,2\}$-effective.

3.2 From *SIEM* to essentially single-valued cores

Now we study the conditions that the existence of an $SIEM$ imposes over cores. We start introducing a richness assumption over $\mathcal{D}$.

Assumption 1. *For each $i \in N$, $R_i \in \mathcal{D}_i$, and $a \in \mathcal{A}^f$ such that aR_ie, there exists $\widetilde{R}_i \in \mathcal{D}_i$ such that*

(i) for each $b \in \mathcal{A}^f$, $bR_ia \Leftrightarrow b\widetilde{R}_ia$ and $aR_ib \Leftrightarrow a\widetilde{R}_ib$;

(ii) for each $b \in \mathcal{A}^f$, $aP_ib \Leftrightarrow a\widetilde{P}_ib$ and $a\widetilde{R}_ie\widetilde{P}_ib$.

If Assumption 1 holds, for every preference R_i and every individually rational allocation a, there exists a preference relation $\widetilde{R}_i$ which lifts e just below a, maintaining the relative ranking of all other allocations with respect to a, and leaving e strictly above all other elements of the strict lower contour set of a. For example, the domain of strict preference profiles satisfies Assumption 1. We start considering the correspondence of efficient and individually rational allocations, $\mathscr{E}\mathscr{I}$. We prove that under Assumption 1, the existence of an $SIEM$ implies that this correspondence is essentially single-valued.

Theorem 1. *If an $SIEM$ exists and Assumption 1 holds, then $\mathscr{E}\mathscr{I}$ is essentially single-valued.*

Proof. Let $R = (R_i)_{i \in N} \in \mathcal{D}$ and let $a \in \mathscr{E}\mathscr{I}(R)$. For all $i \in N$, consider preferences $\widetilde{R}_i \in \mathcal{D}_i$ satisfying (i) and (ii) of Assumption 1, with respect to R_i. Thus $a \in \mathscr{E}\mathscr{I}\left(\widetilde{R}\right)$. Let Γ be an $SIEM$. We prove $\Gamma\left(\widetilde{R}\right)\widetilde{I}_i a$ for all $i \in N$. Let $b \in \mathscr{E}\mathscr{I}\left(\widetilde{R}\right)$. From the definition of $\widetilde{R}$, it follows that $b\widetilde{R}_i a$ for all $i \in N$. We next prove by contradiction $b\widetilde{I}_i a$ for all $i \in N$. Assume there exists $j \in N$ such that $b\widetilde{P}_j a$. Thus, a is not efficient under $\widetilde{R}$ which contradicts $a \in \mathscr{E}\mathscr{I}\left(\widetilde{R}\right)$. Thus $a\widetilde{R}_i b$ for all $i \in N$. The efficiency of b implies $a\widetilde{I}_i b$ for all $i \in N$. Since $\Gamma\left(\widetilde{R}\right) \in \mathscr{E}\mathscr{I}\left(\widetilde{R}\right)$, $\Gamma\left(\widetilde{R}\right)\widetilde{I}_i a$ for all $i \in N$.
Next we prove $\Gamma(R)\, I_i a$ for all $i \in N$. Let $T \subseteq N$ be a set of maximal cardinality such that for all $i \in N \setminus T$, $\Gamma\left(\widetilde{R}_{-T}, R_T\right)\widetilde{I}_i a$ and, for all $i \in T$, $\Gamma\left(\widetilde{R}_{-T}, R_T\right) I_i a$. In order to prove the claim, we will prove, by contradiction, that $T = N$. Assume $T \subsetneq N$ and let $k \notin T$. Set $S = T \cup \{k\}$. By strategy-proofness, for all $j \in S$, $\Gamma\left(\widetilde{R}_{(N\setminus S)\cup\{j\}}, R_{S\setminus\{j\}}\right)\widetilde{R}_j \Gamma\left(\widetilde{R}_{-S}, R_S\right)$ and $\Gamma\left(\widetilde{R}_{-S}, R_S\right) R_j \Gamma\left(\widetilde{R}_{(N\setminus S)\cup\{j\}}, R_{S\setminus\{j\}}\right)$. From the previous part of the proof and the definition of $\widetilde{R}$, we have $\Gamma\left(\widetilde{R}_{-S}, R_S\right) I_j \Gamma\left(\widetilde{R}_{(N\setminus S)\cup\{j\}}, R_{S\setminus\{j\}}\right) I_j a$ for all $j \in N \setminus S$.
Let $b \in \mathscr{E}\mathscr{I}\left(\widetilde{R}_{-S}, R_S\right)$ with $bI_j a$ for all $j \in S$. From the first part of the proof and the definition of $\widetilde{R}$, $b\widetilde{I}_j a$ for all $j \in N \setminus S$. Thus, for all $i \in N \setminus S$, $\Gamma\left(\widetilde{R}_{-S}, R_S\right)\widetilde{I}_i a$ and, for all $i \in S$, $\Gamma\left(\widetilde{R}_{-S}, R_S\right) I_i a$ which contradicts the maximality of T. Then $\Gamma(R) I_i a$ for all $i \in N$. Since a is an arbitrary element of $\mathscr{E}\mathscr{I}(R)$, the transitivity of the preferences implies all allocations in $\mathscr{E}\mathscr{I}(R)$ are indifferent to all agents. It follows that $\Gamma(R)\, I_i a$ for all $i \in N$ and all $R \in D$ such that $\mathscr{E}\mathscr{I}(R) \neq \emptyset$. ■

The claim of Theorem 1 extends to all cores.

Corollary 1. *Assume there exists an $SIEM$ and Assumption 1 holds, then all core correspondences are essentially single-valued on $\mathcal{D}$.*

The result does not imply that every $SIEM$ is a selection of the core.[11] Even if an $SIEM$ exists, it can pick allocations that are vulnerable to coalitional manipulation.

[11] A selection of a correspondence $\mathscr{C} : \mathcal{D} \rightrightarrows \mathcal{A}^f$, is a function $\Gamma : \mathcal{D} \to \mathcal{A}^f$ such that $\Gamma(R) \in \mathscr{C}(R)$ for all $R \in \mathcal{D}$.

Example 2. *Let* $N = \{1,2,3,4\}$ *and* $e = (e_1, e_2, e_3, e_4)$. *Let* $a = (e_1, e_2, e_4, e_3)$, $b = (e_2, e_1, e_3, e_4)$, $c = (e_2, e_1, e_4, e_3)$, *and* $\mathcal{A}^f = \{a, b, c, e\}$. *Consider the following preferences:*

$$M : bP_i aI_i eP_i c; \qquad Q : bP_i eP_i aP_i c; \qquad S : aI_i eP_i cP_i b.$$

Let $\mathcal{D}_1 = \mathcal{D}_2 = \{M, Q\}$, *let* $\mathcal{D}_3 = \mathcal{D}_4 = \{S\}$. *Let* $\mathcal{D} = \prod_{i=1}^{4} \mathcal{D}_i$. *Assumption 1 is satisfied. Consider the SIEM* Γ *defined by* $\Gamma(R) = e$ *for all* $R \in \mathcal{D}$. *From Theorem 1, the core correspondence is essentially single-valued on* $\mathcal{D}$.
For all $x, y \in \mathcal{A}^f$, $T \subseteq N$, *let* $c^M(x, y, T)$ *defined as follows:* $c^M(y, T)(k) = y(k)$ *for all* $k \in T$ *and* $c^M(y, T)(k) = (x(k) \setminus e(T)) \cup (x(T) \cap e(k))$ *for all* $k \notin T$. *Consider the following expectations that we call myopic (Mumcu and Saglam, 2010):* $\Theta_i^M(x, y, T, R) = \left\{c^M(y, T)\right\}$ *if* $\bigcup_{i \in T} y(i) = \bigcup_{i \in T} e_i$ *and* $c^M(y, T) \in \mathcal{A}^f$; $\Theta_i^M(x, y, T, R) = \{x\}$ *otherwise, for all* $i \in T$. *Let* $R^* = (M, M, S, S)$. *We have* $\mathscr{C}^M(R^*) = \{a\}$ *then* $\Gamma(R^*) \notin \mathscr{C}^M(R^*)$.[12]

In Example 2, under preference profile $R^* = (M, M, S, S)$ allocations a and e are indifferent for all agents. However, $\Theta_i^M(a, b, \{1,2\}, R^*) = \{c\}$, but $\Theta_i^M(e, b, \{1,2\}, R^*) = \{b\}$. Thus a belongs to the myopic core bout e doesn't. The following assumption rules out this case.

Assumption 2. *Let* $a, b, c \in \mathcal{A}^f$, $R \in \mathcal{D}$ *and* $T \subseteq N$. *Then,* $aI_i b$ *for all* $i \in N$ $\Longrightarrow \Theta_i(a, c, T, R) = \Theta_i(b, c, T, R)$.

Assume coalition T tries to modify two allocations, a and b, that are equivalent for all members of T, into a third allocation, c. If Assumption 2 holds, the coalition members expect the same allocations regardless of the initial allocation. Assumption 2 is not satisfied by myopic expectations, but it is satisfied, for example, by all expectations independent of the initial allocation, such as the optimistic, the individually rational, the prudent, and the efficient expectations.

Proposition 2. *Assume there exists an SIEM. If Assumptions 1 and 2 hold,* $\Gamma(R) \in \mathscr{C}(R)$ *for all* $R \in \mathcal{D}$ *such that* $\mathscr{C}(R) \neq \emptyset$.

[12] Notice that $\mathscr{C}^O(R) = \mathscr{C}^{IR}(R) = \emptyset$ for all $R \in \mathcal{D}$, while the prudent core correspondence is nonempty and essentially single-valued: $\mathscr{C}^P(R^*) = \{a, e\}$ and $\mathscr{C}^P(R) = \{e\}$ for all $R \in \mathcal{D} \setminus R^*$.

Proof. Let $\Gamma : \mathcal{D} \to \mathcal{A}^f$ be an $SIEM$. Assume Assumption 1 and 2 hold and $\mathscr{C}(R) \neq \emptyset$. We prove by contradiction $\Gamma(R) \in \mathscr{C}(R)$. Assume coalition T blocks $\Gamma(R)$ under preference profile R. Let $a \in \mathscr{C}(R)$. From Theorem 1, $aI_i\Gamma(R)$ for all $i \in N$, because $\Gamma(R) \in \mathscr{E}\mathscr{I}(R)$. From Assumption 2, $\Theta_i(\Gamma(R), b, T, R) = \Theta_i(a, b, T, R)$ for all $b \in \mathcal{A}^f$. Thus, T blocks a as well, which yields a contradiction. ■

Similarly to our Theorem 1, Corollary 1, and Proposition 2, Sönmez (1999) and Ehlers (2018) prove that if an $SIEM$, Γ, exists then the optimistic core and the individually rational core are essentially single-valued and that Γ is a selection of the optimistic core and the individually rational core, respectively. Our results concern a larger family of cores but do not imply Sönmez (1999) and Ehlers (2018) results. Indeed, they employ a different set of assumptions, assumptions A and B. Assumption A requires that, for each $i \in N$, $R \in \mathcal{D}$, and $a \in \mathcal{A}^f$, $aI_ie \iff a(i) = e_i$. It states that an agent who keeps her endowment is not affected by any externality. Thus, it is not compatible with environments with externalities and strict preferences. Assumption B replaces condition (ii) in Assumption 1 with the weaker $(ii)'$: for each $b \in \mathcal{A}^f$, $aP_ib \Leftrightarrow a\widetilde{P}_ib$ and $a\widetilde{R}_ie\widetilde{R}_ib$. Together, A and B are not implied nor imply Assumption 1. Also, under A and B, Theorem 1, Corollary 1, and Proposition 2, do not hold.

Example 3. *Let* $N = \{1, 2, 3, 4\}$ *and* $e = (e_1, e_2, e_3, e_4)$. *Let* $a = (e_3, e_4, e_1, e_2)$, $b = (e_2, e_1, e_3, e_4)$, $c = (e_2, e_1, e_4, e_3)$, *and* $\mathcal{A}^f = \{a, b, c, e\}$. *Consider the following preferences:*

$$M : bP_iaP_icP_ie; \qquad Q : bP_iaP_ieP_ic; \qquad S : bP_ieP_iaP_ic; \qquad U : aP_ibI_ieP_ic.$$

Let $\mathcal{D}_1 = \mathcal{D}_2 = \{M, Q, S\}$, *let* $\mathcal{D}_3 = \mathcal{D}_4 = \{U\}$. *Let* $\mathcal{D} = \prod_{i=1}^4 \mathcal{D}_i$. *Assumptions* A *and* B *hold. The function which assigns* b *to all preference profiles in* $\mathcal{D}$ *is an* $SIEM$. *The prudent core is* $\{a, b\}$ *if agent* 1 *and agent* 2 *have preferences different from* S. *Thus, it is not essentially single-valued.*

3.3 From essentially single-valued cores to *SIEM*

Before introducing the main results, we define a richness condition about the preference domain (see Takamiya (2003)).

Assumption 3. *Let $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$. Let $i \in N$, $R_i \in \mathcal{D}_i$, and $a, b \in \mathcal{A}^f$ be such that aP_ib. Then for all $R_i' \in \mathcal{D}_i$, there exist $\widetilde{R}_i \in \mathcal{D}_i$ such that*

(i) $L^*(b, R_i) \subseteq L^*(b, \widetilde{R}_i)$, *and* $L(b, R_i) \subseteq L(b, \widetilde{R}_i)$; *and*

(ii) $L^*(a, R_i') \subseteq L^*(a, \widetilde{R}_i)$, *and* $L(a, R_i') \subseteq L(a, \widetilde{R}_i)$.

Assumption 3 requires that, if i prefers allocation a to allocation b, then, for each R_i', domain $\mathcal{D}_i$ contains another preference profile $\widetilde{R}_i$ which is a mixture of R_i and R_i': (i) b improves in i's ranking (and no allocation below b reaches it) moving from R_i to $\widetilde{R}_i$;(ii) a improves in i's ranking (and no allocation below a reaches it) moving from R_i' to $\widetilde{R}_i$.

We say that expectations are **monotonic** if, for all $R, R' \in \mathcal{D}$, for all $a \in \mathcal{A}^f$ such that $L(a, R_i) \subseteq L(a, R_i')$ and $L^*(a, R_i) \subseteq L^*(a, R_i')$ for all $i \in N$, then $\Theta_i(a, b, T, R) \cap L(a, R_i) \subseteq \Theta_i(a, b, T, R') \cap L(a, R_i')$ and $\Theta_i(a, b, T, R) \cap L^*(a, R_i) \subseteq \Theta_i(a, b, T, R') \cap L^*(a, R_i')$ for all $i \in N$, $b \in \mathcal{A}^f$, and $T \subseteq N$.

If expectations are monotonic and a improves in everybody's ranking and does not (even weakly) fall in anybody's ranking when preferences change from R to R', it is less likely that any coalition blocks a under R' than under R.

Lemma 1. *Let expectations be monotonic and let $R, R' \in \mathcal{D}$. If $a \in \mathscr{C}(R)$ and $L(a, R_i) \subseteq L(a, R_i')$ and $L^*(a, R_i) \subseteq L^*(a, R_i')$ for all $i \in N$, then $a \in \mathscr{C}(R')$.*

The proof of the result follows directly from the definitions of monotonic expectations and of a core. It thus omitted. Thus, under monotonic expectations, the cores are "almost monotonic" (see Sanver (2006)). All expectations that do not depend on agents' preference profiles, such as the efficient, the prudent, the individually rational, and the optimistic expectations, are monotonic (see also the examples in Appendix A).

Definition 2. *A correspondence* $\mathscr{S} : \mathcal{D} \rightrightarrows \mathcal{A}^f$ *is* ***externally stable*** *if, for all* $a \in \mathscr{I}(R) \setminus \mathscr{S}(R)$*, there exists* $T \subseteq N$ *which blocks* a *announcing some* $b \in \mathcal{A}^f$ *such that* $\mathscr{S}(R) \cap \Theta_i(a, b, T, R) \neq \emptyset$ *for all* $i \in T$ *for all* $R \in \mathcal{D}$*.*

If $\mathscr{S}$ is externally stable, every individually rational allocation outside $\mathscr{S}(R)$ is blocked by a coalition in which everyone expects that an allocation within $\mathscr{S}(R)$ can be attained. Next, we show that if the core correspondence is non-empty and essentially singled valued, all its selections are $SIEM$ if the core is externally stable or the preference domain satisfies Assumption 3.

Theorem 2. *Let* $\mathscr{C}$ *be nonempty and essentially single-valued on* $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$*.*

- (i) *If* $\mathscr{C}$ *is externally stable, any selection of* $\mathscr{C}$ *is an efficient, individually rational, and weakly coalitional strategy-proof mechanism;*
- (ii) *if expectations are monotonic and* $\mathcal{D}$ *satisfies Assumption 3 any selection of* $\mathscr{C}$ *is an efficient, individually rational, and weakly coalitional strategy-proof mechanism.*

Proof. Let $\mathscr{C}$ be nonempty and essentially single-valued. Let $\Gamma : \mathcal{D} \to \mathcal{A}^f$ be a selection of the core correspondence. By definition, Γ is individually rational and efficient.
We prove by contradiction that Γ is weakly coalitionally strategy-proof. Assume there exists $R \in \mathcal{D}$, $T \subseteq N$, and $R'_T \in \prod_{i \in T} \mathcal{D}_i$ such that $\Gamma(R'_T, R_{-T}) \, P_i \, \Gamma(R)$ for all $i \in T$. Since $\mathscr{C}$ is essentially single-valued, we have $\Gamma(R'_T, R_{-T}) \notin \mathscr{C}(R)$. Notice $\Gamma(R'_T, R_{-T}) \in \mathscr{I}(R)$.

- (i) Let $\mathscr{C}$ be externally stable. There exists a coalition $U \subseteq N$ which blocks $\Gamma(R'_T, R_{-T})$ announcing $b \in \mathcal{A}^f$ such that, for each $i \in U$, $\mathscr{C}(R) \cap \Theta_i(\Gamma(R'_T, R_{-T}), b, U, R) \neq \emptyset$. Thus, for all $i \in U$ and there exists $c \in \mathscr{C}(R) \cap \Theta_i(\Gamma(R'_T, R_{-T}), b, U, R)$, $c I_i \Gamma(R)$, because $\mathscr{C}$ is essentially single-valued. In particular, $\Gamma(R) \, R_i \, \Gamma(R'_T, R_{-T})$ for all $i \in U$. Since $\Gamma(R'_T, R_{-T}) \; P_i \, \Gamma(R)$ for all $i \in T$, it follows that $T \cap U = \emptyset$. This implies that U blocks $\Gamma(R'_T, R_{-T})$ when preferences are (R'_T, R_{-T}), which yields a contradiction since $\Gamma(R'_T, R_{-T})$ belongs to $\mathscr{C}(R'_T, R_{-T})$.

(*ii*) Assume expectations are monotonic and Assumption 3 holds. For each $i \in T$, let $\widetilde{R}_i \in \mathcal{D}_i$ as defined in Assumption 3, given $a = \Gamma(R'_T, R_{-T})$ and $b = \Gamma(R)$.

Next, we prove by contradiction that $\Gamma(R) \in \mathscr{C}(\widetilde{R}_T, R_{-T})$. Assume coalition $S \subseteq N$ blocks $\Gamma(R)$ when the preference profile is $(\widetilde{R}_T, R_{-T})$. Monotonicity implies $\Theta_i(\Gamma(R), c, S, R) \cap L_i(\Gamma(R), R) \subseteq \Theta_i\left(\Gamma(R), c, S, \left(\widetilde{R}_T, R_{-T}\right)\right) \cap L_i\left(\Gamma(R), \left(\widetilde{R}_T, R_{-T}\right)\right)$ for all $c \in \mathcal{A}^f$ and all $i \in S$. Then S blocks $\Gamma(R)$ when the preference profile is R which contradicts that $\Gamma(R) \in \mathscr{C}(R)$.

Since $\mathscr{C}$ is essentially single-valued, $\Gamma(\widetilde{R}_T, R_{-T})\widetilde{I}_i\Gamma(R)$ for all $i \in T$, and $\Gamma(\widetilde{R}_T, R_{-T})I_i\Gamma(R)$ for all $i \in N \setminus T$. Similarly, from Assumption 3, $\Gamma(\widetilde{R}_T, R_{-T})\widetilde{I}_i\Gamma(R'_T, R_{-T})$ for all $i \in T$, and $\Gamma(\widetilde{R}_T, R_{-T})I_i\Gamma(R'_T, R_{-T})$ for all $i \in N \setminus T$. Then, $\Gamma(R)\widetilde{I}_i\Gamma(R'_T, R_{-T})$ for all $i \in T$, and $\Gamma(R)I_i\Gamma(R'_T, R_{-T})$ for all $i \in N \setminus T$. Since $\Gamma(R'_T, R_{-T})P_i\Gamma(R)$ for all $i \in T$, N blocks $\Gamma(R)$ announcing $\Gamma(R'_T, R_{-T})$, which contradicts $\Gamma(R) \in \mathscr{C}(R)$.

■

Theorem 2 extends Proposition 1 in Sönmez (1999), Theorem 1 in Takamiya (2003), and Proposition 2 in Ehlers (2018), who focus on optimistic expectations, and individually rational expectation.

If Assumption 1 holds and an $SIEM$ exists, the correspondence of efficient and individually rational allocations is essentially single-valued. Moreover, in finite environments, the efficient and individually rational correspondence is externally stable and never empty.

Lemma 2. *If $\mathcal{A}^f$ is finite*

(*i*) $\mathscr{E}\mathscr{I}(R) \neq \emptyset$ *for all* $R \in \mathcal{D}$.

(*ii*) $\mathscr{E}\mathscr{I}$ *is externally stable.*

Thus, $\mathscr{E}\mathscr{I}$ completely characterizes the set of $SIEM$ under Assumption 1.

Proposition 3. *Let $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$ and let $\mathcal{A}^f$ be finite. If Assumption 1 holds, an $SIEM$ exists if and only if $\mathscr{E}\mathscr{I}$ is essentially single-valued on $\mathcal{D}$. Further, if $\mathscr{E}\mathscr{I}$ is essentially single-valued, the set of $SIEM$ coincides with the set of selections from $\mathscr{E}\mathscr{I}$.*

The correspondence of efficient and individually rational allocations also satisfies Assumption 2. Therefore, the result follows directly from Theorems 1 and 2. Additionally, Assumptions A and B imply Assumption 1 if the unique allocation in which somebody keeps their endowment is e. Thus, Proposition 3 implies Corollary 1 in Ehlers (2018) about global trades.

The statement of Theorem 2 stands on two assumptions about the core $\mathscr{C}$: the non-emptiness and the essentially single-valuedness. Indeed, the first condition is more likely to be satisfied by larger cores, such as the prudent core or the efficient and individually rational correspondence. The second condition, instead, is more likely to be satisfied by smaller cores, such as the optimistic core. Thus, satisfying both requirements generates a trade-off for the application of Theorem 2. The next example applies Theorem 2 to the correspondence of efficient and individually rational allocations, which is single-valued. Instead, smaller cores, such as the optimistic and the individually rational cores, are empty.[13]

Example 4. *Let* $N = \{1, 2, 3, 4\}$ *and* $e = (e_1, e_2, e_3, e_4)$. *Let* $a = (e_2, e_3, e_4, e_1)$, $b = (e_2, e_1, e_3, e_4)$, $c = (e_2, e_1, e_4, e_3)$, *and* $d = (e_1, e_2, e_4, e_3)$. *The set of feasible allocations is* $\mathcal{A}^f = \{a, b, c, d, e\}$. *The preferences are:*

$$M : bP_iaP_ieP_icP_id; \qquad Q : bP_ieP_iaP_icP_id; \qquad S : aP_ieP_idP_icP_ib.$$

Let $\mathcal{D}_1 = \mathcal{D}_2 = \{M, Q\}$, *let* $\mathcal{D}_3 = \mathcal{D}_4 = \{S\}$. $\mathcal{D} = \prod_{i\in N} \mathcal{D}_i$. *Assumptions 3 is satisfied. We have* $\mathscr{C}^O (R) = \mathscr{C}^{IR} (R) = \emptyset$ *for all* $R \in \mathcal{D}$. *However,* $\mathscr{E}\mathscr{I} (M, M, S, S) = \{a\}$ *and* $\mathscr{E}\mathscr{I} (R) = \{e\}$ *for all* $R \in \mathcal{D} \setminus (M, M, S, S)$. *Proposition 3 applies. Thus, the mechanism that selects the unique element of* $\mathscr{E}\mathscr{I} (R)$ *for all* $R \in \mathcal{D}$ *is the unique* $SIEM$ *with domain* $\mathcal{D}$.

In Example 4, Assumption 1 holds. If this condition is not satisfied, Theorem 1 no longer applies. Thus, having an essentially single-valued efficient and individually rational correspondence is no longer necessary for an $SIEM$ to exist. The next example presents a

[13]Thus, also Proposition 2 in Ehlers (2018), Proposition 1 in Sönmez (1999), or Theorem 1 in Takamiya (2003) do not apply.

finite environment in which we can apply Theorem 2 to the optimistic and the individually rational core since they are singletons. However, the prudent is not essentially single-valued.

Example 5. *Consider the environment of Example 3. Assumption 1 is not satisfied, and Assumption 3 is satisfied. The optimistic and the individually rational cores coincide with* $\{b\}$ *for all preference profiles in* $\mathcal{D}$*. This, fact, via Theorem 2 provides an alternative proof of the fact that the constant function which equals* b *for all preference profiles in* $\mathcal{D}$ *is an* $SIEM$*. However, the prudent core is not essentially single-valued.*

4 Applications

4.1 Coalition Formation Problems

We employ the notation of Remark 1, (i). We consider problems in which agents care first about the coalition they belong to and then about the other coalitions. We prove that, under this condition, if $\mathcal{A}^f$ is single-lapping, the core is a singleton, and Assumption 3 holds; thus, we can apply Theorem 2 and prove that the social choice function picking the unique core allocation is strategy-proof.

We now introduce some additional notation. Given a coalition formation problem and $a \in \mathcal{A}^C$, let $A(a,i) = \{j \in N : \omega_{ji} \in a(i)\} \cup \{i\}$, the coalition to which agent i is assigned by a. Let $\mathcal{A}^f \subseteq \mathcal{A}^C$. We say that agent i has **block preferences** if she cares primarily about the coalition she belongs to. Formally, if for all $a, a', a'' \in \mathcal{A}^f$ such that $A(a,i) = A(a',i)$ and $A(a,i) \neq A(a'',i)$, we have $a \not{I}_i a''$ and $aP_i a'' \Rightarrow a'P_i a''$, and $a''P_i a \Rightarrow a''P_i a'$. Let $\mathcal{D}_i^B$ be the set of block preferences for agent i. A block preference profile $R = (R)_{i \in N}$ is a profile of preferences in which each agent has block preferences. Let $\mathcal{D}^B = \prod_{i \in N} \mathcal{D}_i^B$. Let $\mathcal{D}_i = \{R \in \mathcal{D}_i^B : aR_i e \, \forall a \text{ such that } a(i) = e_i\}$, be the set of block preferences in which e is the least preferred allocation in which agent i keeps her individual endowment. Finally, let $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$.

Lemma 3. *Domains* $\mathcal{D}$ *and* $\mathcal{D}^B$ *satisfy Assumption 3.*

It follows from Theorem 2, that an essentially single-valued core is a sufficient condition for the existence of an $SIEM$. We now introduce a restriction on the set of feasible allocations (see Pápai (2004) and Fonseca-Mairena and Triossi (2023)). Let $\Xi^f = \{T \subseteq N : \exists a \in \mathcal{A}^f, i \in N \text{ such that } T = A(a,i)\}$ be the set of feasible coalitions. The set of feasible allocations $\mathcal{A}^f$ is **single-lapping** if

(i) for all $T, T' \in \Xi^f$ such that $T \neq T'$, $|T \cap T'| \leq 1$, and

(ii) for all $\{T_1, \ldots, T_m\} \subseteq \Xi^f$ such that $m \geq 3$ and for all $l = 1, \ldots, m$, $|T_l \cap T_{l+1}| \geq 1$, where we let $T_{m+1} := T_1$, there exists $i \in N$ such that for all $l = 1, \ldots, m$, $T_l \cap T_{l+1} = \{i\}$.

If $\mathcal{A}^f$ is single-lapping, two coalitions share at most one agent, and any sequence of overlapping coalitions shares exactly the same agent. In this case, the set of unblocked allocations is a singleton.

Lemma 4. *Let $R \in \mathcal{D}^B$. If $\mathcal{A}^f$ is single-lapping, the set of unblocked allocations under prudent expectations is a singleton.*

The result follows from Lemmata 9 and 10 in the Appendix. In Lemma 9, we prove that the stable set and the core coincide under block preferences and prudent expectations. It amounts to prove that we can restrict our attention to blocking where the deviating members form a unique coalition. In Lemma 10, we prove that, under block preferences and a single-lapping $\mathcal{A}^f$, the stable set is a singleton, employing the characterization of singleton stable sets by Pápai (2004) for environments in which the agents only care about the coalition they belong to. If the preference profile belongs to $\mathcal{D}^B$, the set of unblocked allocations under prudent expectations may fail to satisfy individual rationality. Thus, we restrict our attention to $\mathcal{D} \subseteq \mathcal{D}^B$. From Theorem 2 and Lemma 4, we have:

Proposition 4. *If $\mathcal{A}^f$ is single-lapping, the unique nonempty selection of the prudent core correspondence, $\mathscr{C}^P : \mathcal{D} \rightrightarrows \mathcal{A}^f$ is strategy-proof, and thus it is an $SIEM$.*

The result does not follow from either Proposition 1 in Sönmez (1999) nor Proposition 2 in Ehlers (2018). Indeed, the optimistic core and the individually rational core can be empty under the hypothesis of Proposition 4 as proven in the following example.

Example 6. *Let* $N = \{1,2,3,4\}$*. Let* $\mathcal{A}^f = \{a,b,c,d,e\}$ *where* $\omega_{31} \in a(1)$*,* $\omega_{42} \in a(2)$*,* $\omega_{31} \in b(1)$*,* $b(2) = e_2$*,* $b(4) = e_4$*,* $\omega_{41} \in c(1)$*,* $c(2) = e_2$*,* $c(3) = e_3$*,* $d(1) = e_1$*,* $\omega_{42} \in d(2)$*,* $d(3) = e_3$*.* $\mathcal{A}^f$ *is single-lapping. The preference profile* $R \in \mathcal{D}$ *is*

$$R_1 : bP_1aP_1cP_1dP_1e; \qquad R_2 : bP_2aP_2cP_2dP_2e;$$

$$R_3 : aP_3dP_3bP_3cP_3e; \qquad R_4 : aP_4dP_4bP_4cP_4e.$$

We have $\mathscr{C}^P(R) = \{a\}$ *while* $\mathscr{C}^O(R) = \mathscr{C}^{IR}(R) = \emptyset$*.*

4.2 Marriage markets

We employ the notation of Remark 1, (iv). Let $\widetilde{\mathcal{D}} = \{P \in \mathcal{P}^{|N|} : P_w = P_{w'}\ \forall w, w' \in W\}$. Domain $\widetilde{\mathcal{D}}$ includes strict preference profiles in which all women have the same preferences. This domain satisfies Assumption 1 but not Assumption A. Consequently, Theorem 1 in Sönmez (1999) and Theorem 1 in Ehlers (2018) do not apply.

Proposition 5. *In a marriage market in which* $|W| \geq 2$ *and* $|M| \geq 2$*, there is no* $SIEM$*,* $\Gamma : \prod_{i \in N} \mathcal{S}_i \to \mathcal{A}^f$*, in which* $\widetilde{\mathcal{D}} \subseteq \prod_{i \in N} \mathcal{S}_i$ *and* $\prod_{i \in N} \mathcal{S}_i$ *satisfies Assumption* 1*.*

Next, employing Theorem 2 we prove the existence of an $SIEM$ in a restricted domain of preferences with externalities. Similarly to the previous section, we say that agent i has **block preferences**, if she cares primarily about her pair. Formally, if for all $a, a', a'' \in \mathcal{A}^f$ such that $a(i) = a'(i)$ and $a(i) \neq a''(i)$, we have $a \not{I}_i a''$ and $aP_ia'' \Rightarrow a'P_ia''$ and $a''P_ia \Rightarrow a''P_ia'$. Let $\widehat{\mathcal{D}}_i^B$ be the set of block preference profiles for agent i and let $\widehat{\mathcal{D}}^B = \prod_{i \in N} \widehat{\mathcal{D}}_i^B$.

Let $w \in W$ and $R_w \in \widehat{\mathcal{D}}_w^B$, and define the following order over $M \cup \{w\}$. For all $x, y \in M \cup \{w\}$, let $x\overline{P}_w y$ if there exist $a, b \in \mathcal{A}^f$ such that $a(w) = x$ and $b(w) = y$ such that aP_wb. For all $m \in M$ and $R_m \in \widehat{\mathcal{D}}_m^B$, define $\overline{P}_m$ similarly.

Given a profile $Q_W = (Q_w)_{w \in W}$ in which Q_w is a linear order over $M \cup \{w\}$, we say that Q_W has a cycle if there is a list of men and women "in a circle" in which every listed woman prefers the man on his clockwise side to the man on his counterclockwise side and finds both acceptable. Formally, a cycle (of length $T+1$) in the preferences of the women is given by $w_0, w_1, ..., w_T$ such that $w_t \neq w_{t+1}$ for $t = 0, ..., T$ and distinct $m_0, m_1, ..., m_T$ such that (i) $m_T Q_{w_T} m_{T-1} ... m_1 Q_{w_1} m_0 Q_{w_0} m_T$; (ii) for every $t \geq 1$, $m_t Q_{w_t} w_t$ and $m_{t-1} Q_{w_t} w_t$.[14] We say that Q_W is acyclic if it has not any cycle.

Given $R_W \in \left(\widehat{\mathcal{D}}^B_w\right)_{w \in W}$, $R_W = (R_w)_{w \in W}$, we say that it is acyclic if $\left(\overline{P}_w\right)_{w \in W}$ is acyclic. Let $R'_W \in \left(\widehat{\mathcal{D}}^B_w\right)_{w \in W}$ be acyclic and such that, for all $w \in W$, $a(w) = w$, $\Rightarrow aR'_w e$. Let $\widehat{\mathcal{D}} = \{P \in \widehat{\mathcal{D}}^B : R_w = R'_w \; \forall w \in W, \text{for all } m \in M, a(m) = m \Rightarrow aR_m e\}$.

Lemma 5. *Domain $\widehat{\mathcal{D}}$ satisfies Assumption 3.*

Lemma 6. *If $R \in \widehat{\mathcal{D}}$, $\left|\mathscr{C}^P(R)\right| = 1$.*

Applying Lemmata 5, 6, and Theorem 2 yields:

Proposition 6. *The unique selection of the prudent core correspondence $\mathscr{C}^P : \widehat{\mathcal{D}} \rightrightarrows \mathcal{A}^f$ is an SIEM.*

The result does not follow from either Proposition 1 in Sönmez (1999) nor Proposition 2 in Ehlers (2018). Indeed, the optimistic core and the individually rational core can be empty under the hypothesis of Proposition 6 as proven in the following example.

Example 7. *Let $N = \{w_1, w_2, m_1, m_2\}$. Let $\mathcal{A}^f = \{a, b, c, d, e\}$ where $a(w_1) = \{m_1\}$, $a(w_2) = \{m_2\}$, $b(w_1) = \{m_1\}$, $b(w_2) = e_{w_2}$, $c(w_1) = \{m_2\}$, $c(w_2) = e_{w_2}$, $d(w_1) = e_{w_1}$, $d(w_2) = \{m_2\}$. Let $R \in \widehat{\mathcal{D}}$ be*

$$R_{w_1} : bP_{w_1}aP_{w_1}cP_{w_1}dP_{w_1}e; \qquad R_{w_2} : bP_{w_2}aP_{w_2}cP_{w_2}dP_{w_2}e;$$

$$R_{m_1} : aP_{m_1}dP_{m_1}bP_{m_1}cP_{m_1}e; \qquad R_{m_2} : aP_{m_2}dP_{m_2}bP_{m_2}cP_{m_2}e.$$

We have $\mathscr{C}^P(R) = \{a\}$ while $\mathscr{C}^O(R) = \mathscr{C}^{IR}(R) = \emptyset$.

[14]From this point forward, indices are considered modulo $T+1$.

5 Comments and extensions

In models without externalities, individual rationality is also an individual participation constraint. In environments with externalities, the individual rationality condition is a welfare condition and/or a normative prescription, which ensures that the final allocation will make nobody worse off with respect to her initial situation. Relaxing the individual rationality constraint, any unblocked allocation satisfies, in particular, an individual participation constraint: no agent wants to opt-out. This constraint guarantees that each agent weakly prefers an unblocked allocation to an allocation she expects can be attained if she keeps her individual endowment. We call an allocation with this property **participative**, a concept that depends on expectations. In the case of optimistic expectations, this amounts to individual rationality. In the case of prudent expectations, the constraint only requires that each agent weakly prefers the allocation to at least one feasible allocation in which she keeps her individual endowment.

We relax the individual rationality requirement and consider strategy-proof and efficient mechanisms that only satisfy the individual participation constraint (under Assumption A in Sönmez, 1999 and Ehlers, 2018, the participation constraint and individual rationality are equivalent). We call these mechanisms strategy-proof, participative, and efficient mechanisms or $SPEM$. Instead of the core analyzed previously, we consider the set of allocations that are not blocked by any coalition. We call this set the participative core, denoted by $\mathscr{C}^*(R)$. The core is always a subset of the participative core. The findings of Theorem 1 do not extend to $SPEM$.

Example 8. *Consider the set of allocations and preferences of Example 3. In addition, consider the following preferences*

$$V : aP_ieP_ibP_ic.$$

Let $\mathcal{D}_1 = \mathcal{D}_2 = \{M, Q, S\}$*, let* $\mathcal{D}_3 = \mathcal{D}_4 = \{U, V\}$*. Let* $\mathcal{D} = \prod_{i=1}^{4} \mathcal{D}_i$*. Assumption 1 is satisfied. The constant mechanism equal to* b *is an* $SPEM$ *but not an* $SIEM$ *under prudent expectations. However, the prudent participative core is* $\{a, b\}$ *if* $R =$

(M, M, U, U). Then, the prudent core is not essentially single-valued.

On the other hand, Theorem 2 extends to participative cores. The next condition slightly strengthens the definition of an externally stable correspondence.

Definition 3. *A correspondence $\mathscr{S} : \mathcal{D} \rightrightarrows \mathcal{A}^f$ is **strongly externally stable** if, for all $a \in \mathcal{A}^f \setminus \mathscr{S}(R)$, there exists $T \subseteq N$ which blocks a announcing some $b \in \mathcal{A}^f$ such that $\mathscr{S}(R) \cap \Theta_i(a, b, T, R) \neq \emptyset$ for all $i \in T$ for all $R \in \mathcal{D}$.*

If $\mathscr{S}$ is strongly externally stable, every allocation outside $\mathscr{S}(R)$ is blocked by a coalition in which everyone expects that an allocation within $\mathscr{S}(R)$ can be attained.[15]

Proposition 7. *Let $\mathscr{C}^*$ be nonempty and essentially single-valued on $\mathcal{D} = \prod_{i \in N} \mathcal{D}_i$.*

- *(i) If $\mathscr{C}^*$ is strongly externally stable, any selection of $\mathscr{C}^*$ is an efficient, participative, and weakly coalitional strategy-proof mechanism;*
- *(ii) if expectations are monotonic and $\mathcal{D}$ satisfies Assumption 3 any selection of $\mathscr{C}^*$ is an efficient, participative, and weakly coalitional strategy-proof mechanism.*

The proof of the Proposition 7 follows the same argument as the proof of Theorem 2 and is thus omitted. The application of Theorem 2 to coalition formation problems proves the existence of an $SIEM$ in the case of a single-lapping $\mathcal{A}^f$, if the preference domain of consists of all block preferences in which each agent ranks the initial endowment at the end of its block, which is if the preference profiles belong to $\mathcal{D}$. In this domain, $\mathscr{C}^P$ coincides with the prudent participative core. However, the prudent participative core is not, in general, individually rational if the initial endowment is not at the end of its block for some agents, which on $\mathcal{D}^B \setminus \mathcal{D}$. If we relax the assumption of individual rationality, Proposition 7 allows us to prove the existence of the result of Proposition 4 to the set of all block preferences.

Corollary 2. *If $\mathcal{A}^f$ is single-lapping, the unique nonempty selection of the prudent core correspondence, $\mathscr{C}^{*P} : \mathcal{D}^B \rightrightarrows \mathcal{A}^f$ is strategy-proof, and it is thus an $SPEM$.*

The result follows from Lemmata 3 and 4, and Proposition 7. A similar extension can be developed for the case of marriage markets following Subsection 4.2.

[15]External stability imposes conditions only on individually rational allocations outside $\mathscr{S}$.

6 Conclusions

We have introduced a model of cores based on the expectations of the agents belonging to a deviating coalition about the behavior of the agents outside the coalition. Explicitly modeling expectations provide a framework encompassing several previously introduced models that deal with coalitional stability under externalities. Within this framework, if the preference domain is rich enough, the existence of an $SIEM$ implies that the correspondence of efficient and individually rational allocations is essentially single-valued, thus providing a strong necessary condition. On the other hand, if any of the cores are essentially single-valued, any selection of that core is an $SIEM$. Thus, in finite environments with a rich enough preference domain, the correspondence of efficient and individually rational allocations completely characterizes all $SIEM$. We also present applications to coalition formation problems and marriage markets and discuss the effects of relaxing the individually rational constraint in favor of a weaker individual participation constraint.

Appendix A: Others specifications of expectations

Let $a, b \in \mathcal{A}^f$, $T \subseteq N$, and $R \in \mathcal{D}$. We next present examples of expectations.

(i) Consider the following expectations, that we call weakly optimistic:

$$\Theta_i^{WO}(a,b,T,R) = \begin{cases} \{b\} \text{ if } \bigcup_{i\in T} b(i) = \bigcup_{i\in T} e_i, \text{ and } a(i) \neq b(i) \text{ for some } i \in T; \\ \{a\}, \text{ otherwise} \end{cases}$$

for all $i \in T$. Under weakly optimistic expectations, a coalition blocks an allocation a only if it blocks a under optimistic expectations employing an allocation in which at least one of its agents receives objects different from the ones she has in a. We call weakly optimistic core, denoted by $\mathscr{C}^{WO}(R)$, the core resulting from weakly optimistic expectations. The weakly optimistic core coincides with the ω-core studied in Hong and Park (2022). Weakly optimistic expectations are monotonic but not proper.

(ii) Consider the following expectations, that we call complementary efficient. $\Theta_i^{PE}(a,b,T,R) = \{c \in \Theta_i^P(a,b,T,R) : d \in \Theta_i^P(a,b,T,R), dP_jc \text{ for some } j \in T^c \Rightarrow cP_kd \text{ for some } k \in T^c\}$ for all $i \in T$.[16] If expectations are complementary efficient and coalition T announces b, each agent in T expects that the agents outside T will not choose an inefficient allocation once $(b(i))_{i\in T}$ is taken as given. We call complementary efficient core, denoted by $\mathscr{C}^{PE}$, the core resulting from complementary efficient expectations.

Complementary efficient expectations are proper, monotonic, and vary with R.

(iii) Consider an allocation problem, $\mathscr{A} = (N, e, \mathcal{A}^f, (R_i, \Theta_i)_{i\in N})$ in which $(R_i)_{i\in N} \in \mathcal{D}$. Let $\emptyset \nsubseteq T \subseteq N$. For all $b \in \mathcal{A}^f$, let $\mathcal{A}^f(T^c, b) = \{\left((b(j))_{j\in T}, (x(i))_{i\in T^c}\right) : \left((b(j))_{j\in T}, (x(i))_{i\in T^c}\right) \in \mathcal{A}^f\}$. Let $\Omega \in \{O, P\}$. Expectations are defined recursively based on the cardinality of N. Let $a, b \in \mathcal{A}^f$

- for $|N| = 1$, let $\Theta_i^{R\Omega}(a,b,T,R) = \Theta_i^{\Omega}(a,b,T,R)$ for all T.

[16] We denote by X^c the complement of set X.

– Assume $\Theta_i^{R\Omega}(a,b,T,R)$ has been defined for all allocations problems such that $1 \leq |N| \leq n$. Let $T \subsetneq N$, $T \neq \emptyset$.
Let $\mathscr{A}(T^c, b) = \left(T^c, (e_i)_{i \in T^c}, \mathcal{A}^f(T^c, b), \left(R_i, \Theta_i^{R\Omega}\right)_{i \in T^c}\right)$.

$$\Theta_i^{R\Omega}(a,b,T,R) = \begin{cases} \mathscr{C}^{\Omega}(\mathscr{A}(T^c,b)) \text{ if } \bigcup_{i \in T} b(i) = \bigcup_{i \in T} e_i \text{ and } \mathscr{C}^{\Omega}(\mathscr{A}(T^c,b)) \neq \emptyset; \\ \{a\}, \text{ otherwise.} \end{cases}$$

Finally, let $\Theta_i^{R\Omega}(a,b,N,R) = \{b\}$.

We call expectations Θ_i^{RO} and Θ_i^{RP}, recursively prudent and recursively optimistic, respectively. The cores generated by these expectations are called recursively prudent core and recursively optimistic core, respectively, and are denoted by $\mathscr{C}^{RP}$ and $\mathscr{C}^{RO}$. They correspond to the recursive cores defined in Kóczy (2007) for partition function games.

The next example shows that recursive expectations are not monotonic.

Example 9. *Let $N = \{1,2,3,4,5\}$ and let*

$$a = (e_1, e_2, e_4, e_5, e_3); \quad b = (e_2, e_1, e_3, e_4, e_5); \quad c = (e_2, e_1, e_3, e_5, e_4);$$

$$d = (e_2, e_1, e_4, e_5, e_3); \quad e = (e_1, e_2, e_3, e_4, e_5); \quad f = (e_2, e_1, e_5, e_3, e_4).$$

Let $\mathcal{A}^f = \{a,b,c,d,e,f\}$. Let the preferences of the agents be such that

$$R_1 : bP_1cI_1dP_1aI_1eP_1f; \quad R_2 : bP_2cP_2dP_2fP_2aI_2e;$$

$$R_3 : cP_3dP_3fP_3\ldots; \quad R_4 =: dP_4cP_4fP_4bP_4\ldots; \quad R_5 =: dP_4cP_4fP_4bP_4\ldots.$$

Let $R = (R_1, R_2, R_3, R_4, R_5)$. Allocations $a, b, e,$ and f are blocked by $\{3,4,5\}$ announcing c. Therefore, any core is a subset of $\{c,d\}$. We have $\Theta_i^{RO}(c,b,\{1,2\},R) = \Theta_i^{RO}(d,b,\{1,2\},R) = \{c\}$. If agents 1 and 2 have recursive optimistic expectations, they block d announcing b but do not block the allocation c. We have $\mathscr{C}^{RO}(R) = \{c\}$. If agents *have recursively prudent expectations* $\Theta_i^{RP}(c, b, \{1,2\}, R) = \Theta_i^{RP}(d, b, \{1,2\}, R) = \{c, d\}$. *We have* $\mathscr{C}^{RP}(R) = \{c, d\}$.

Next, let the preferences of agents 4 and 5 be such that

$$R'_4 =: dP'_4 bP'_4 cP'_4 fP'_4 \ldots; \qquad R'_5 =: dP'_5 bP'_5 cP'_5 fP'_5 \ldots.$$

Let $R' = (R_1, R_2, R_3, R'_4, R'_5)$. *We have* $L(d, R_i) \subseteq L(d, R'_i)$ *and* $L^*(d, R_i) \subseteq L^*(d, R'_i)$ *for all* $i \in N$. *However* $\Theta_i^{RO}(d, c, \{1,2,3\}, R) \cap L(d, R_i) = \{c\} \nsubseteq \Theta_i^{RO}(c, d, \{1,2,3\}, R') \cap L(d, R'_i) = \{b\}$ *and* $\Theta_i^{RP}(d, c, \{1,2,3\}, R) \cap L(d, R_i) = \{c\} \nsubseteq \Theta_i^{RP}(c, d, \{1,2,3\}, R') \cap L(d, R'_i) = \{b\}$. *Then, recursive expectations are not monotonic.*

Appendix B: Proof of the Results in Subsection 3.1

Lemma 7. *Let* $\mathscr{A} = (N, e, \mathcal{A}^f, R, (\Theta_i)_{i\in N})$ *and* $\mathscr{A}' = (N, e, \mathcal{A}^f, R, (\Theta'_i)_{i\in N})$. *If* $\Theta_i(a, b, T, R) \neq \emptyset$ *and* $\Theta_i(a, b, T, R) \subseteq \Theta'_i(a, b, T, R)$ *for all* $a, b \in \mathcal{A}^f$, *for all* $T \subseteq N$, *and for all* $i \in N$, *then* $\mathscr{C}(\mathscr{A}) \subseteq \mathscr{C}'(\mathscr{A}')$.

Proof. Assume coalition T blocks $a \in \mathcal{A}^f$ under expectations $(\Theta'_i)_{i\in N}$ announcing $b \in \mathcal{A}^f$. Since $\Theta_i(a, b, T, R) \subseteq \Theta'_i(a, b, T, R)$ for all $i \in N$, coalition T blocks a under $(\Theta_i)_{i\in N}$ announcing b. Thus, $\mathscr{C}(\mathscr{A}) \subseteq \mathscr{C}'(\mathscr{A}')$. ■

Lemma 8. *If the expectations* $(\Theta_i)_{i\in N}$ *are proper then* $\Theta_i \subseteq \Theta_i^P$ *for all* $i \in N$.

Proof. Let $(\Theta_i)_{i\in N}$ be a proper expectation. Then $(\Theta_i)_{i\in N}$ is T-effective for all $T \subseteq N$. Let $a, b \in \mathcal{A}^f$, $T \subseteq N$, $i \in N$, and $R \in \mathcal{D}$ such that $\bigcup_{j\in T} b(j) = \bigcup_{j\in T} e_j$, then $\Theta_i(a, b, T, R) \neq \emptyset$ and $\Theta_i(a, b, T, R) \subseteq \{c : c(i) = b(i), \text{ for all } i \in T\} = \Theta_i^P(a, b, T, R)$. ■

Appendix C: Proof of the Results in Subsection 3.3

Proof of Lemma 2. Define the following order on $\mathcal{A}^f$: $a \succ b$ if aR_ib for all $i \in N$ and there exists $j \in N$ such that aP_jb. Observe that allocation a is efficient if there exists no b such that $b \succ a$. Notice that $\succ$ is transitive.

We prove that for every $a \in \mathscr{I}(R) \setminus \mathscr{E}\mathscr{I}(R)$, there exists $b \in \mathscr{E}\mathscr{I}(R)$ such that $b \succ a$.

Let $\{a_i\}_{i\in\mathbb{N}}$ defined as follows. Since $a \in \mathscr{I}(R)$. a_1 such that $a_1 \succ a$. Notice that a_1 is individually rational. For all $i \geq 1$, let $a_{i+1} \succ a_i$ if $a_i \in \mathscr{I}(R)$ and let $a_{i+1} = a_i$, otherwise. Since $\mathcal{A}^f$ is finite, there exists n such that $a_{n+i} = a_n$ for all i. Let $b = a_n$. We have $b \in \mathscr{E}\mathscr{I}(R)$ and $b \succ a$, which proves (ii).

Let $a = e$. If $e \notin \mathscr{E}(R)$, there exists $b \in \mathscr{E}\mathscr{I}(R)$ such that $b \succ e$, which proves (i). ■

Appendix D: Proof of the Results in Subsection 4.1

Proof of Lemma 3. We prove the claim for $\mathcal{D}$. The proof for $\mathcal{D}^B$ is identical. Let $i \in N$. Let R_i be a block preference and let $b, c \in \mathcal{A}^f$ such that cP_ib. We prove that for all $R_i' \in \mathcal{D}_i$ there exists $R_i^* \in \mathcal{D}_i$ such that Assumption 3 holds. Consider the following cases:

(i) Let $cP_i'b$. Let R_i^* be any block preference such that $L(b, R_i^*) = L(b, R_i) \cup L(b, R_i')$, $L^*(b, R_i^*) = L^*(b, R_i) \cup L^*(b, R_i')$, $L(c, R_i^*) = L(c, R_i) \cup L(c, R_i')$, and $L^*(c, R_i^*) = L^*(c, R_i) \cup L^*(c, R_i')$. Such a R_i^* exists because R_i and R_i' are block preferences. Then $L(b, R_i) \subseteq L(b, R_i^*)$, $L^*(b, R_i) \subseteq L^*(b, R_i^*)$, $L(c, R_i') \subseteq L(c, R_i^*)$, and $L^*(c, R_i') \subseteq L^*(c, R_i^*)$. Thus, $R_i^* \in \mathcal{D}_i$ satisfies Assumption 3.

(ii) Let $bP_i'c$. Let R_i^* be any block preference such that $L(b, R_i^*) = L(b, R_i) \cup L(b, R_i')$, $L^*(b, R_i^*) = L^*(b, R_i) \cup L^*(b, R_i')$, $L(c, R_i^*) = L(c, R_i')$, and $L^*(c, R_i^*) = L^*(c, R_i')$.[17] Such a R_i^* exists because R_i and R_i' are block preferences. Then $L(b, R_i) \subseteq L(b, R_i^*)$, $L^*(b, R_i) \subseteq L^*(b, R_i^*)$, $L(c, R_i') = L(c, R_i^*)$, and $L^*(c, R_i') = L^*(c, R_i^*)$. Thus, $R_i^* \in \mathcal{D}_i$ satisfies Assumption 3.

■

Next, we define stable allocations. For each $T \subseteq N$ let $\mathcal{A}^f(T) = \{a \in \mathcal{A}^f : A(a, i) = T \text{ for all } i \in T\}$. $\mathcal{A}^f(T)$ is the set of allocations in which the members of T form a coalition. An allocation a is stable if there is no coalition $T \in \Xi^f$ and $b \in \mathcal{A}^f(T)$ such that cR_ia for all $c \in \Theta_i^P(a, b, T, R)$ for all $i \in T$ and cP_ja for all $c \in \Theta_j^P(a, b, T, R)$ for some $j \in T$. The stable set is the set of stable allocations and it is denoted by

[17] Notice that, if $bI_i'c$, then bI_i^*c and, if $bP_i'c$, then bP_i^*c.

$\mathscr{S}(R)$.[18] In general, a stable allocation can fail to be in the prudent core because the agents in blocking coalition T block can split in several disjoint sub-coalitions $T_1, T_2, ..., T_k$, $T_1 \cup T_2 \cup ... \cup T_k = T$. Let $\mathscr{C}^{*P}(R)$ be the set of unblocked allocations under prudent expectations when the preference profile is R.

Lemma 9. *If $R \in \mathcal{D}^B$ then $\mathscr{S}(R) = \mathscr{C}^{*P}(R)$.*

Proof. Let $R \in \mathcal{D}^B$. The proof of the claim is divided in two parts.

(i) We prove by contradiction $\mathscr{C}^{*P}(R) \subseteq \mathscr{S}(R)$. Let $a \in \mathscr{C}^{*P}(R)$. Assume that $a \notin \mathscr{S}(R)$. Then there exists $T \in \Xi^f$ and $b \in \mathcal{A}^f(T)$ such that cR_ia for all $c \in \Theta_i^P(a,b,T,R)$ for all $i \in T$ and cP_ja for all $c \in \Theta_j^P(a,b,T,R)$ for some $j \in T$. From the definition of core, $a \notin \mathscr{C}^{*P}(R)$, which yields a contradiction.

(ii) We prove by contradiction $\mathscr{S}(R) \subseteq \mathscr{C}^{*P}(R)$. Let $a \in \mathscr{S}(R)$. By the definition of $\mathcal{D}^B$, it follows $a \in \mathscr{I}(R)$. Assume that $a \notin \mathscr{C}^{*P}(R)$. Then $T \subseteq \Xi^f$ and $b \in \mathcal{A}^f$ such that cR_ia for all $c \in \Theta_i^P(a,b,T,R)$ for all $i \in T$ and cP_ja for all $c \in \Theta_j^P(a,b,T,R)$ for some $j \in T$. If $b \in \mathcal{A}^f(T)$, then, from the definition of stable set, $a \notin \mathscr{S}(R)$ which yields a contradiction. If $b \notin \mathcal{A}^f(T)$, let $T' = A(b,j)$. Then, there exists $b \in \mathcal{A}^f(T')$, $j \in T'$. Since $b \in \Theta_i^P(a,b,T',R) \cap \Theta_i^P(a,b,T,R)$ for all $i \in T'$, bR_ia for all $i \in T'$ and cP_ja. Since $R \in \mathcal{D}^B$, cR_ia for all $c \in \Theta_i^P(a,b,T',R)$ for all $i \in T'$ and cP_ja for all $c \in \Theta_j^P(a,b,T',R)$. Thus, from the definition of stable set $a \notin \mathscr{S}(R)$, which yields a contradiction.

■

Lemma 10. *Let $R \in \mathcal{D}^B$. If $\mathcal{A}^f$ is single-lapping $|\mathscr{S}(R)| = 1$.*

Proof. First, we modify R into a preference profile without externalities, $\widetilde{R}$. For all $a \in \mathcal{A}^f$, let $a^{w,R_i} \in \mathcal{A}^f$ be on of the worst allocation, according to R_i, in which agent i keeps the same coalition as in a, that is $A\left(a^{w,R_i}, i\right) = A(a,i)$ and $a'R_ia^{w,R_i}$ for all a' such that $A(a',i) = A(a,i)$. Let $\widetilde{R} \in \widehat{\mathcal{D}}^B$ be such that:

(a) for all $a, a' \in \mathcal{A}^f$ such that $A(a',i) = A(a,i)$, $a\widetilde{I}_ia'$;

[18] Notice that if $b \in \mathcal{A}^f(T)$, then $\Theta_i^P(a,b,T,R) \subseteq \mathcal{A}^f(T)$.

(b) for all $a, a' \in \mathcal{A}^f$ such that $A(a', i) \neq A(a, i)$, $a\widetilde{P}_i a'$ if $a^{w,R_i} P_i a'^{w,R_i}$.

Prefence profile $\widetilde{R}$ is well defined because $R \in \widehat{\mathcal{D}}$. From Theorem 1 in Pápai (2004), $\left|\mathscr{S}(\widetilde{R})\right| = 1$. Next we prove that $\mathscr{S}\left(\widetilde{R}\right) = \mathscr{S}(R)$.

(*i*) We prove by contradiction $\mathscr{S}\left(\widetilde{R}\right) \subseteq \mathscr{S}(R)$. Assume there exists $a \in \mathscr{S}\left(\widetilde{R}\right) \setminus \mathscr{S}(R)$. Then, there exists $T \in \Xi^f$ and $b \in \mathcal{A}^f(T)$ such that $cR_i a$ for all $c \in \Theta_i^P(a, b, T, R)$ for all $i \in T$ and $cP_j a$ for all $c \in \Theta_j^P(a, b, T, R)$ for some $j \in T$. From the definition of $\widetilde{R}$, $c\widetilde{R}_i a$ for all $c \in \Theta_i^P\left(a, b, T, \widetilde{R}\right)$ for all $i \in T$ and $c\widetilde{P}_j a$ for all $c \in \Theta_j^P\left(a, b, T, \widetilde{R}\right)$ for some $j \in T$. Thus $a \notin \mathscr{S}\left(\widetilde{R}\right)$ which yields a contradiction.

(*ii*) We prove by contradiction $\mathscr{S}(R) \subseteq \mathscr{S}\left(\widetilde{R}\right)$. Assume there exists $a \in \mathscr{S}(R) \setminus \mathscr{S}\left(\widetilde{R}\right)$. Then, there exists $T \in \Xi^f$ and $b \in \mathcal{A}^f(T)$ such that $cR_i a$ for all $c \in \Theta_i^P\left(a, b, T, \widetilde{R}\right)$ for all $i \in T$ and $c\widetilde{P}_j a$ for all $c \in \Theta_j^P\left(a, b, T, \widetilde{R}\right)$ for some $j \in T$. From the definition of $\widetilde{R}$and since $c^{w,R_i} \in \Theta_i^P\left(a, b, T, \widetilde{R}\right)$, $cR_i a$ for all $c \in \Theta_i^P(a, b, T, R)$ for all $i \in T$ and $cP_j a$ for all $c \in \Theta_j^P(a, b, T, R)$. Thus $a \notin \mathscr{S}(R)$ which yields a contradiction.

■

Appendix E: Proof of the Results in Subsection 4.2

Proof of Proposition 5. Let $\widetilde{M} = \{m_1, m_2\} \subseteq M$ and let $\widetilde{W} = \{w_1, w_2\} \subseteq W$. Let $a, b, c, d, f, g \in \mathcal{A}^f$ be such that: $a(w_i) = m_i$ for $i \in \{1, 2\}$; $b(w_1) = m_1$, $b(w_2) = w_2$, and $b(m_2) = m_2$; $c(w_i) = m_j$ for $i, j \in \{1, 2\}$, $i \neq j$; $d(w_1) = w_1$, $d(w_2) = m_1$, and $d(m_2) = m_2$; $f(w_1) = w_1$, $f(w_2) = m_2$, and $f(m_1) = m_1$; $g(w_1) = m_2$, $g(w_2) = w_2$, and $g(m_1) = m_1$.

Let $P^* \in \widetilde{\mathcal{D}}$ be such that

$$bP^*aP^*cP^*dP^*fP^*gP^* \ldots P^*e;$$

let P_{m_1} and P_{m_2} satisfy

$$bP_{m_1}aP_{m_1}cP_{m_1}fP_{m_1}dP_{m_1}gP_{m_1} \ldots P_{m_1}e;$$

$$cP_{m_2}aP_{m_2}bP_{m_2}dP_{m_2}fP_{m_2}gP_{m_2}\dots P_{m_2}e.$$

For each $m \in M$ let P_m be such that xP_me for all $x \in \mathcal{A}^f \setminus \{e\}$. Let $P_w = P^*$ for all $w \in W$ and let $P = (P_w, P_m)_{w\in W, m\in M} \in \widetilde{\mathcal{D}}$. We have $\mathscr{C}^P(P) = \{a, b\}$. Thus, the prudent core is not essentially single-valued. Then, the claim follows from Theorem 1. ■

Proof of Lemma 5. Let $i \in N$. Let R_i be a block preference and let $b, c \in \mathcal{A}^f$ such that cP_ib. We prove that for all $R'_i \in \widehat{\mathcal{D}}_i$ there exists $R^*_i \in \widehat{\mathcal{D}}_i$ such that Assumption 3 holds. Consider the following cases:

(i) Let cP'_ib. Let R^*_i be any block preference such that $L(b, R^*_i) = L(b, R_i) \cup L(b, R'_i)$, $L^*(b, R^*_i) = L^*(b, R_i) \cup L^*(b, R'_i)$, $L(c, R^*_i) = L(c, R_i) \cup L(c, R'_i)$, and $L^*(c, R^*_i) = L^*(c, R_i) \cup L^*(c, R'_i)$. Such a R^*_i exists because R_i and R'_i are block preferences. Then $L(b, R_i) \subseteq L(b, R^*_i)$, $L^*(b, R_i) \subseteq L^*(b, R^*_i)$, $L(c, R'_i) \subseteq L(c, R^*_i)$, and $L^*(c, R'_i) \subseteq L^*(c, R^*_i)$. Thus, $R^*_i \in \widehat{\mathcal{D}}_i$ satisfies Assumption 3.

(ii) Let bP'_ic. Let R^*_i be any block preference such that $L(b, R^*_i) = L(b, R_i) \cup L(b, R'_i)$, $L^*(b, R^*_i) = L^*(b, R_i) \cup L^*(b, R'_i)$, $L(c, R^*_i) = L(c, R'_i)$, and $L^*(c, R^*_i) = L^*(c, R'_i)$.[19] Such a R^*_i exists because R_i and R'_i are block preferences. Then $L(b, R_i) \subseteq L(b, R^*_i)$, $L^*(b, R_i) \subseteq L^*(b, R^*_i)$, $L(c, R'_i) = L(c, R^*_i)$, and $L^*(c, R'_i) = L^*(c, R^*_i)$. Thus, $R^*_i \in \widehat{\mathcal{D}}_i$ satisfies Assumption 3.

■

Next, we define pairwise stable allocations. Allocation a is pairwise stable if it is not blocked by a coalition T with $|T| \leq 2$ under prudent expectations.

Lemma 11. *If $R \in \widehat{\mathcal{D}}^B$ then $\mathscr{S}(R) = \mathscr{C}^P(R)$.*

Proof. Let $R \in \widehat{\mathcal{D}}^B$. The fact that $\mathscr{C}^P(R) \subseteq \mathscr{S}(R)$, follows from the definition of pairwise stable allocations.

Now, we prove by contradiction $\mathscr{S}(R) \subseteq \mathscr{C}^P(R)$. Let $a \in \mathscr{S}(R)$. By the definition of $\widehat{\mathcal{D}}^B$, it follows $a \in \mathscr{I}(R)$. Assume that $a \notin \mathscr{C}^P(R)$. Then, there exists a coalition

[19]Notice that, if bI'_ic, then bI^*_ic and, if bP'_ic, then bP^*_ic.

T and $b \in \mathcal{A}^f$ such that cR_ia for all $c \in \Theta_i^P(a,b,T,R)$ for all $i \in T$ and cP_ja for all $c \in \Theta_j^P(a,b,T,R)$ for some $j \in T$. Let $T' = \{i \in T : b(i) \neq a(i)\}$. Let $i \in N \cap T'$. Without loss of generality, assume there exists $w \in W \cap T'$. If $b(w) = w$, then, by the definition of block preferences, it follows that $a \notin \mathscr{E}\mathscr{I}(R)$, which yields a contradiction. Otherwise $b(w) = m \in M \cap T'$. Thus, the definition of block preferences implies $\{m, w\}$ blocks a under prudent expectations, which yields a contradiction. ■

Lemma 12. *Let* $P \in \left(\widehat{\mathcal{D}}^B\right)^{|N|}$ *such that* P_W *is acyclic. Then,* $\left|\mathscr{C}^P(R)\right| = 1$.

Proof. From Fonseca-Mairena and Triossi (2019), $|\mathscr{S}(P)|$ coincides with the set of pairwise stable allocations of $\overline{P}$. Since $\overline{P}_W$ is acyclic, from Romero-Medina and Triossi (2023), $\left|\mathscr{S}\left(\overline{P}\right)\right| = 1$. Then the claim follows from Lemma 11. ■

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