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Abstract

This study examines the dynamics of capital stocks distributed among several nodes, representing different sites of production and connected via a weighted, directed network.
The network represents the externalities or spillovers that the production in each node generates on the capital stock of other nodes. A regulator decides to designate some of the nodes for the production of a consumption good to maximise a cumulative utility from consumption.
It is demonstrated how the optimal strategies and stocks depend on the productivity of the resource sites and the structure of the connections between the sites. The best locations to host production of the consumption good are identified per the model's parameters and correspond to the least central (in the sense of eigenvector centrality) nodes of a suitably redefined network that combines both flows between nodes and the nodes' productivity.

Keywords: Capital allocation, Production externalities, Network spillovers, Economic centrality measures.

JEL Codes: C61, D62, O41, R12.

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GROWTH MODELS WITH EXTERNALITIES ON NETWORKS

GIORGIO FABBRI*, SILVIA FAGGIAN†, AND GIUSEPPE FRENI‡

Abstract. This study examines the dynamics of capital stocks distributed among several nodes, representing different sites of production and connected via a weighted, directed network. The network represents the externalities or spillovers that the production in each node generates on the capital stock of other nodes. A regulator decides to designate some of the nodes for the production of a consumption good to maximize a cumulative utility from consumption. It is demonstrated how the optimal strategies and stocks depend on the productivity of the resource sites and the structure of the connections between the sites. The best locations to host production of the consumption good are identified per the model's parameters and correspond to the least central (in the sense of eigenvector centrality) nodes of a suitably redefined network that combines both flows between nodes and the nodes' productivity.

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1. Introduction

There is a growing interest in the literature in the study of the economic effects of heterogeneous interactions of different entities. Multisector growth models with externalities (e.g., Benhabib et al., 2000), metapopulation models of interconnected natural resources (e.g., Sanchirico and Wilen, 2005) and network models of various kinds (e.g., Ballester et al., 2006, Elliott and Golub, 2019) are examples of this trend. In this paper, we take a network perspective in studying a multisector growth model with externalities.

We consider a growth model where production is distributed among several locations, different for productivity, and connected by the fact that production in one engenders positive externalities on the production of the others. A single agent aims to localize the production of a consumption good, in order to maximize the sum of the sites' utility from consumption.

This work develops a simple dynamic model where the n ($n \geq 0$) nodes of a weighted, directed network represent the n sites where the capital stock is accumulated, and the weight on the edges between two nodes represent the externalities of production in one node on the production in the other. Specifically, it aims to show how the structure of the network and other parameters of the system affect the agent decision in the choice of one or more nodes/locations for the production of the consumption good.

As the main contribution to the literature, the study shows that when the agent is sufficiently "patient", in the generalized growth theory sense that their rate of discount is close to a critical discount rate,[1] and the network is strongly connected, the (possibly not unique) optimal closed-loop strategies are linear in the stock, exhibiting an analytic formula for such strategies (Theorems 1 and 3). Moreover, at optimum, independently of the assignment, the different site stocks are evaluated via a constant common vector of relative prices that proves to be the eigenvector centrality of another related network that combines the spillover effect and the sites' net rates of growth. The effect of these two forces are jointly captured by the adjacency matrix of such modified network, that is the sum of the adjacency matrix of the original network and the diagonal matrix of net productivities of sites.[2] Moreover, it is proven that the best allocation of the production of the consumption good is at the most peripheral node(s), namely that (those) with the least centrality. For initial stocks in a cone contained in the positive orthant (and characterized by means of eventually exponentially positive matrices, as in Noutsos and Tsatsomeros, 2008), such allocation is placed immediately at the most peripheral node. For other initial capital stocks, the allocation in the most peripheral node is best in the long run, and initially may have to be placed otherwise, notably when the initial capital stock in the most peripheral nodes is small. Furthermore, if the least eigencentrality is unique, the optimal control is unique, at least when the initial stock belongs to such cone.

The model is further extended to enclose transportation costs (Theorem 3), whose effect is to modify the eigencentrality and the associated hierarchy of nodes.

This model is closely related to the one developed in Fabbri et al. (2024), and has been briefly introduced in Fabbri et al. (2022) with the difference that in such works it

[1]See e.g., McFadden, 1973 for a discussion of critical discount rates in optimal growth theory.

[2]The resulting matrix may well have negative terms on the principal diagonal, although the other entries remain nonnegative - i.e. it is a Metzler matrix - hence the associated network could be referred to as a "signed" network. Nonetheless, the fact that its adjacency matrix is Metzler helps preserving several properties, such as the Perron-Frobenius property.

is assumed that several independent players occupy one node of exclusive exploitation and interact in a dynamic game. In that case, the results in Fabbri et al. (2024) apply, providing a divergent outcome, as externalities are not internalized by the different agents. The model is also related to that in the continuous space-time growth models by Boucekkine et al. (2013); Fabbri (2016); Boucekkine et al. (2019) and the discrete space version developed by Calvia et al. (2023). While some of the techniques employed overlap with those found in the aforementioned papers, the economic models feature notable distinctions. In Boucekkine et al. (2013) the model involves strategic iterations for natural resource extraction, in Fabbri (2016) consumption takes place independently at each node[3] (at every node, a portion of the production is not invested, resulting in consumption that takes place exclusively on-site). Furthermore, in the latter cases, explicit results are provided exclusively for the symmetric scenario, involving the Laplacian in continuous time and symmetric matrices in the discrete case. These disparities also manifest themselves in the system's behavioral aspects when examining its asymptotic state in response to variations in agents' preference parameters (as discussed in Section 3.3).

The article is structured as follows: in Section 2 we introduce the model and discuss its mathematical formulation. In Section 3, we present the main results of the article: the introduction of the candidate optimal strategy (Theorem 1) and its admissibility (Theorem 2), followed by a discussion (Subsection 3.3) on the asymptotic properties of the system as the parameters of the agents' preferences vary. Section 4 introduces the extension of the model with transportation costs. Section 5 concludes. Appendix A contains the proofs of statements.

2. The model

We consider capital stocks available in different but interconnected areas that are considered to be sufficiently different from each other (and sufficiently homogeneous in their interior) so as to be described by different parameters. We then analyze a growth model in which the production in an area generates non-negative spillovers on stocks in the others.

[3]This hypothesis can be conceived as an economic model featuring infinitely high transportation costs. The findings presented in Section 4, illustrating the extension of our results to include explicit (iceberg) transportation costs, may be viewed as an intermediary model bridging the gap between the two extremes: one characterized by infinite transportation costs and the other by negligible transportation costs.

Mathematically speaking, we consider a network $\mathcal{G}$ with n nodes – as many as the number of subareas – that we assume to be directed and weighted.

We denote by $K_i(t)$ the capital stock at node i at time t and by y_i the local productions that we assume to linearly depend on the used capital: $y_i = \Gamma_i K_i$, where $\Gamma_i > 0$ is a productivity coefficient. We suppose that production at a node j generates the spillover $b_{ji}y_j = b_{ji}\Gamma_j K_j$ at node i, where b_{ij} are given nonnegative coefficients. They are the weight of the links of our network so that $\mathcal{G}$ will represent the spillover network of our economy that we will suppose to be strongly connected. $B = (b_{ij})$ is the adjacency matrix of $\mathcal{G}$.

The budget constraint at each location i imposes that the augmented production $\Gamma_i K_i(t) + \sum_{j\neq i} b_{ji}\Gamma_j K_j(t)$ is split at each time between the consumption $c_i(t)$ and the investment in the location-specific investment $I_i(t)$. We assume that investments are reversible (i.e., each $I_i(t)$ can be negative). If we suppose that the capital at the location j decays at rates δ_i we get the evolution of the the capital stock at node-i:

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t) = (\Gamma_i - \delta_i)K_i(t) + \sum_{j\neq i} b_{ji}\Gamma_j K_j(t) - c_i(t),$$

where $i, j \in \mathbb{N}$, $1 \leq i, j \leq n$, and in matricial form the system dynamics is given by

$$\begin{cases} \dot{K}(t) = [\Gamma - D + B^\top \Gamma]K(t) - c(t), \quad t \geq 0 \\ K(0) = k \end{cases} \tag{1}$$

where Γ is the diagonal matrix of productivities Γ_i, $B = (b_{ij})$, D is the (also diagonal) matrix of decay rates δ_i, $c(t) = (c_1(t), \cdots, c_n(t))^\top$

We require the capital stocks in every node to be nonnegative, that is

$$K_i(t) \geq 0, \quad \forall i, \forall t \geq 0. \tag{2}$$

Our goal is to identify the nodes where a certain agent, having free access to all, prefers to produce the consumption good or, equivalently, those from which they prefer to draw resources for consumption. We assume that the total consumption of the agent is

$$\sum_{i=1}^{n} c_i(t) = \langle c(t), \mathbf{e}\rangle$$

where $\mathbf{e} = \sum_{i=1}^{n} e_i = (1, 1, \cdots, 1)^\top$, and that they maximize the functional

$$J(c) = \int_0^{+\infty} e^{-\rho t} u\left(\sum_{i=1}^{n} c_i(t)\right) dt = \int_0^{+\infty} e^{-\rho t} u\left(\langle \mathbf{e}, c(t)\rangle\right) dt, \tag{3}$$

with u the utility function

$$u(c) = \ln(c) \quad or \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0,\ \sigma \neq 1$$

(the case of a logarithmic u stands for the case $\sigma = 1$), and $\rho \in \mathbb{R}$ is the discount rate.[4]

In the above model, all externalities are non-negative. However, extensions of the Perron-Frobenius theory to matrices with some negative entries (for example, eventually positive or eventually exponentially positive matrices, see e.g. Farina and Rinaldi, 2000) can be used to extend the analysis to cases in which positive and negative externalities coexist.

Remark 1 Since by assumption the network $\mathcal{G}$ is strongly connected, the matrix B is irreducible. Moreover, by hypothesis, B is non-negative. Since Γ is diagonal with strictly positive values on the diagonal, the same properties hold for ΓB, so that the matrix

$$\Gamma B + \Gamma - D$$

is again irreducible and has non-negative values out of the diagonal (it is a Metzler matrix). This fact has three consequences:

(*i*) $\Gamma B + \Gamma - D$ is again the adjacency matrix of a network (in this case a *signed network* because it might have negative loops)

(*ii*) we can apply to $\Gamma B + \Gamma - D$ the Perron-Frobenius theorem in its strong form (see Bapat and Raghavan, 1997, Theorem 1.4.4 page 17) and conclude that it has a simple (not necessarily positive) real eigenvalue λ, strictly greater than the real parts of the other eigenvalues, and with a unique *positive* associated normalized eigenvector η.

(*iii*) Since $\Gamma B + \Gamma - D$ is irreducible and has non-negative non-diagonal entries, then its transpose matrix

$$B^{\top}\Gamma + \Gamma - D$$

enjoys the same properties and then has a unique positive eigenvector ζ (different from η, in general) associated to the same dominant (i.e. with real part strictly larger than the real part of any other eigenvalue) eigenvalue λ.

[4]The results hold regardless of the sign of ρ. Although a negative discount rate is uncommon in applications, a stream of literature considers "upcounting" (see e.g., Le Van and Vailakis, 2005, Dolmas, 1996, and Rebelo, 1991).

3. Explicit Solutions

We here identify the set of parameters for which there exists an explicit solution. We do so by means of Bellman's Dynamic Programming. The associated Hamilton-Jacobi-Bellman (briefly, HJB) equation is

$$\rho v(k) = H(\nabla v(k)) + \langle \nabla v(k), [(I + B^\top)\Gamma - D]k\rangle \tag{4}$$

where the Hamiltonian $H(p)$ is given by

$$H(p) = \max_{c\geq 0}\left\{u\left(\langle \mathbf{e}, c\rangle\right) - \langle c, p\rangle\right\} = \begin{cases} \dfrac{\sigma}{1-\sigma}\left(\min_i p_i\right)^{1-\frac{1}{\sigma}}, & \sigma \neq 1, \\ -\left[\ln\left(\min_i p_i\right) + 1\right] & \sigma = 1. \end{cases} \tag{5}$$

To prove the last equality, we start by noting that: (a) the above maximum is attained on the boundary except when $p/|p| = \mathbf{e}/|\mathbf{e}|$; (b) we are assuming (no positivity constraint on stocks) the relaxed condition (2); (c) for equal extracted stocks $\langle \mathbf{e}, c\rangle$, the maximum is obtained when $-\sum_{i=1}^n p_i c_i$ is minimized, hence when all the consumption takes place at the node(s) where prices p_i are at their minimum, i.e. at nodes in $N(p)$ where

$$N(p) = \{\ell : p_\ell = \min_i p_i\} \subseteq \{1, 2, ..., n\}.$$

Let c_i^* denote the elements of the maximizing vector c^*. Then

$$-\sum_{i=1}^n p_i c_i^* = -\min_i p_i \sum_{i\in N(p)} c_i^* \equiv -(\min_i p_i) q^*$$

where

$$q^* = \mathrm{argmax}_q\{u(q) - q(\min_i p_i)\} = (u')^{-1}(\min_i p_i)$$

Thus, the candidate optimal consumption is of type

$$c_i^* = 0, \quad \forall i \notin N(p) \quad \text{and} \quad \sum_{i\in N(p)} c_i^* = q^*$$

and (5) readily follows.

3.1. **Optimal Strategies.**

We now provide an explicit solution to the model problem in the assumption that θ, defined as

$$\theta := \begin{cases} \dfrac{\rho - \lambda(1-\sigma)}{\sigma}, & \sigma \neq 1 \\ \rho, & \sigma = 1 \end{cases} \tag{6}$$

is (positive and) small enough, as described in (13). For the sake of simplicity, we initially assume assume that there is a unique minimal (positive) coordinate of the eigenvector η. By possibly renaming the nodes, we can assume $\eta_1 = \min_i \eta_i$. For the extension of the theorem to multiple minimizers, the reader is referred to Theorem 3.

THEOREM 1 (Optimal Strategies) *Assume $\sigma \neq 1$, $\theta > 0$, and that η_1 is the unique minimal coordinate of η. The following facts hold:*

(i) *when admissible, the closed-loop optimal control is c^* such that*

$$c_i^* = 0, \quad \forall i \neq 1 \quad \text{and} \quad c_1^*(t) = \frac{\theta}{\eta_1}\langle K(t), \eta\rangle \tag{7}$$

(ii) *moreover, for all $k \neq 0$, the value function of the problem in section 2 is*

$$v(k) = \frac{\theta^{-\sigma}}{1-\sigma}\eta_1^{\sigma-1}\langle k, \eta\rangle^{1-\sigma}; \tag{8}$$

(iii) *the associated optimal trajectory $K^*(t)$ satisfies*

$$\langle K^*(t), \eta\rangle = \langle k, \eta\rangle e^{(\lambda-\theta)t} \tag{9}$$

Proof. See Appendix A. □

REMARK 2 One can prove that in the case of logarithmic utility we have

$$\theta = \rho, \quad V(k) = \frac{1}{\rho}\left[\ln\left(\frac{\langle k, \eta\rangle \rho}{\eta_1}\right) - 1\right].$$

Next we discuss admissibility of the optimal control described by (7). The following remarks come handy:

(a) The closed-loop equation (briefly, CLE), namely the evolution system associated to the optimal control (7), has the form

$$\dot{K}(t) = AK(t)$$

where A is the matrix

$$A = \Gamma - D + B^\top \Gamma - \frac{\theta}{\eta_1} e_1 \eta^\top. \tag{10}$$

so that the optimal trajectory is given by

$$K^*(t) = e^{tA}k.$$

(b) It is also useful to recall that a matrix A is said *eventually exponentially non negative (positive)*, with exponential index t_0 if

$$e^{tA} \geq 0 \quad (e^{tA} > 0), \quad \forall t \geq t_0. \tag{11}$$

All Metzler matrices are eventually exponentially non-negative with exponential index 0, and viceversa (see Lemma 3.1 in Noutsos and Tsatsomeros, 2008).

Thus condition (13) in the Theorem 2 is equivalent to requiring A eventually exponentially positive with exponential index 0, from which nonnegativity of the trajectory is inferred, for every initial nonnegative stock k.

In the next Theorem we discuss under which assumptions the control described by (7) is admissible in terms of eventual exponential positivity of the matrix A.

Theorem 2 (Admissibility) *Assume that η_1 is the unique minimal coordinate of η. Assume also $\sigma \neq 1$, $\theta > 0$. The optimal control c^* described in by (7) is admissible (and then optimal) in the following two sets of assumptions:*

(i) *if A given by (10) is eventually exponentially positive, with index $t_0 = t_0(A)$ and the initial stock k lies in the cone K, defined by*

$$K := e^{t_0 A}\left(\mathbb{R}^n_+\right). \tag{12}$$

(ii) *if θ satisfies*

$$0 < \theta < \eta_1 \frac{\Gamma_j b_{j1}}{\eta_j}, \qquad \textit{for all } j. \tag{13}$$

and the initial stock k is in the positive orthant $\mathbb{R}^n_+$.

Proof. See Appendix A. □

Remark 3 The assumption of A being eventually exponentially positive clearly yields an implicit bound on the magnitude of θ. We refer the reader to Section 3.3 where we discuss how such implicit condition can be further explicitated.

3.2. **Long-run Stocks.** We now analyze the long-term behavior of the stock, establishing if the stock tends to stabilize over time around certain values at different nodes. Note that for a null extraction, the convergence is toward the direction of the eigenvector ζ associated with the dominant eigenvalue λ. Here, we will explain how the equilibrium extraction reduces the growth rate to $\lambda - \theta$ and modifies the direction of the associated eigenvector to $\hat{\zeta}$.

In the following lemma we establish a relationship between the eigenvectors and eigenvalues of $A^\top$ to those of $\Gamma - D + B^\top \Gamma$ (regardless of whether condition (13) is met).

Lemma 1 *Let η and ζ be respectively the (real) eigenvectors of the matrices $\Gamma B + \Gamma - D$ and its transpose, both associated to the dominant eigenvalue λ, as described in Remark 1.*

(i) *The vector η is an eigenvector of $A^\top$ associated with the eigenvalue $\lambda - \theta$; hence, there exists a real eigenvector $\hat{\zeta}$ of A associated with $\lambda - \theta$. If $\theta > 0$ is small enough then $\lambda - \theta$ is the dominant eigenvalue of both the matrices and $\hat{\zeta}$ is a positive vector.*

(ii) *Consider a basis $\{\zeta, v_2, \ldots, v_n\}$ of generalized eigenvectors of $\Gamma - D + B^\top\Gamma$, associated with the eigenvalues $\{\lambda, \lambda_2, \ldots, \lambda_n\}$. Then $\{\hat{\zeta}, v_2, \ldots, v_n\}$ is a basis of generalized eigenvectors for A associated with eigenvalues $\{\lambda - \theta, \lambda_2, \ldots, \lambda_n\}$. In particular the eigenspace associated to $\lambda - \theta$ has dimension 1.*

Proof. See Appendix A. □

We now establish that, in the long run, the optimal trajectory K^* converges towards the direction of the eigenvector $\hat{\zeta}$ of A or, more precisely, that the *detrended optimal trajectory*

$$Y(t) = e^{-(\lambda-\theta)t} K^*(t)$$

converges towards a multiple of $\hat{\zeta}$, provided θ is small enough.

Proposition 1 *In the assumptions of Theorem 2, and for*

$$0 < \theta < \lambda - \operatorname{Re} \lambda_2,$$

where λ_2 is the eigenvalue with greatest real part among $\{\lambda_2, ..., \lambda_n\}$, the detrended optimal trajectory $Y(t)$ satisfies

$$\lim_{t \to +\infty} Y(t) = \frac{\langle k, \eta \rangle}{\langle \hat{\zeta}, \eta \rangle} \hat{\zeta} \tag{14}$$

3.3. **Bounds on Impatience of the Decision Maker.** We have already noted that the assumption of A being eventually exponentially positive, appearing in Theorem 2(i) and in Proposition 1, conceals an implicit bound on the magnitude of θ, embodying "impatience" of the decision maker. Such interpretation is straightforward for logarithmic utility where $\theta = \rho$ and a small enough θ can be seen as the decision maker being sufficiently patient.

We intend to provide a more explicit bound and try robustness of the results of Theorem 2(i) with respect to changes in the agent preferences in terms of impatience.

Remark 4 It will not be restrictive to limit the analysis to the case of a non-negative matrix A. Indeed the matrix A is a Metzler matrix. If there are negative elements on the main diagonal, we can add to A the matrix aI and for a big enough the resulting matrix $A + aI$ has the same eigenspaces (and generalized eigenspaces) as

A but a spectrum which is shifted by a in the complex plane. For our purposes, which primarily involve understanding the relative ranking of the real parts of the eigenvalues of A and its submatrices, as well as the behaviors of the associated eigenvectors, there is no loss of generality in assuming that A is non-negative.

REMARK 5 We recall the following facts:

(i) a matrix M is said to have *strong Perron–Frobenius property* if its spectral ray $\rho(M)$ is a (positive, real) simple eigenvalue, strictly larger than the norm of all other eigenvalues, and associated to a positive eigenvector; for nonnegative matrices, the condition on maximal norm can be replaced by $\rho(M)$ larger than *the real part* of all other eigenvalues;

(ii) the property (11) is equivalent to the following fact (Theorem 3.3 in Noutsos and Tsatsomeros, 2008):
There exists $a \geq 0$ such that $A + aI$ and $A^\top + aI$ both have the strong Perron–Frobenius property.

Now we analyse what happens when θ grows, starting from a positive level close to zero. Preliminarily, we observe that the feedback described in equation (7) naturally extends to the limiting case of $\theta = 0$ (even though the optimization problem is ill-posed in this scenario), with $c^* = 0$. In this situation, the system evolves with a matrix A defined in (10) coinciding with that of the system without extraction $\Gamma B + \Gamma - D$, implying $\zeta = \hat{\zeta}$. As noted in Remark 1, A is irreducible, and thus (since it is also non-negative) it satisfies the strong Perron-Frobenius property described in Remark 5. Consequently, every detrended trajectory converges to a multiple of ζ, in view of Proposition 1.

When instead θ is strictly positive, Lemma 1 indicates the two phenomena at play for increasing values of θ:

(1) the eigenvector $\hat{\zeta}$ (which is ζ modified by the effect of consumption) may cease to be contained inside the positive orthant; in this case, the trajectory associated to A may bear negative or null components; possibly this fact takes place for θ surpassing a first threshold θ_1;

(2) if θ surpasses the threshold $\theta_2 := \lambda - Re(\lambda_2)$, then $\lambda - \theta$ is no longer the greatest eigenvalue of A; in this case, the trajectories of the system ruled by A no longer converge towards the direction of $\hat{\zeta}$, the system becomes unstable, and condition (11) fails to hold (although a trajectory starting on the direction of $\hat{\zeta}$ may still be optimal).

In what follows we will prove that, at least in the case in which λ_2 is real

$$0 < \theta_1 \leq \theta_2$$

meaning that the stability is lost before the modified dominant eigenvector $\hat{\zeta}$ leaves the positive orthant, and we provide a characterization of θ_1.

LEMMA 2 *We define λ_{22} as the dominant eigenvalue*[5] *of the $(n-1) \times (n-1)$ matrix A_{22}, obtained from A by removing the first row and the first column, and we set $\theta_1 = \lambda - \lambda_{22}$. If $0 < \theta < \theta_1$, then $\hat{\zeta}$ is a positive vector. If $\theta = \theta_1$ then $\hat{\zeta}$ is non-negative with null first component $\hat{\zeta}_1$.*

Proof. See Appendix A. □

The next proposition orders the thresholds θ_1, θ_2 when λ_2 is real.

PROPOSITION 2 *Suppose that λ_2, the second (ordered in terms of greatest real part) eigenvector of $\Gamma - D + B^\top\Gamma$ is real. Then, as long as $0 < \theta < \lambda - \lambda_{22}$, we have that $\lambda_2 < \lambda - \theta$.*

Proof. See Appendix A. □

The above proposition implies $0 < \theta_1 \leq \theta_2$, where $\theta_1 = \lambda - \lambda_{22}$ and $\theta_2 = \lambda - \lambda_2$ Hence, Theorems 2 and Proposition 1 hold both for $0 < \theta < \lambda - \lambda_{22}$.

4. An Extension of the Model with Transportation Costs

We now assume that the intertemporal utility takes into account iceberg-type transportation costs $\beta_i \in [0, 1)$ ($\beta_i = 0$ meaning no loss of consumption goods during transportation, while $\beta_i = 1$ would mean a complete loss), namely

$$J(c) = \int_0^{+\infty} e^{-\rho t} u \left(\sum_{i=1}^{n} (1 - \beta_i) c_i(t) \right) dt, \tag{15}$$

where $\beta = (\beta_1, \cdots, \beta_n)^\top$. At the same time we remove the simplifying assumption that $\eta_1 = \min_i \eta_i$ with η_1 being the unique minimal coordinate of η. In such case, the Hamiltonian function becomes

$$\tilde{H}(p) = \max_{c \geq 0} \left\{ u \left(\langle \mathbf{e}, (I - \mathcal{B})c \rangle \right) - \langle c, p \rangle \right\},$$

[5]The matrix A_{22} is non-negative (see Remark 4), we can then apply the weak form of the Perron-Frobenius Theorem and obtain that A_{22} has a non-negative real eigenvalue, larger (or equal) than the real part of any other eigenvalue of A_{22}.

where

$$I - \mathcal{B} = \begin{pmatrix} 1-\beta_1 & 0 & \cdots & 0 \\ 0 & 1-\beta_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1-\beta_n \end{pmatrix}$$

and noted that $\langle c, p \rangle = \langle (I-\mathcal{B})^{-1} p, (I-\mathcal{B}) c \rangle$, one has

$$\tilde{H}(p) = H((I-\mathcal{B})^{-1}p) = \begin{cases} \dfrac{\sigma}{1-\sigma} \left(\min_i \dfrac{p_i}{1-\beta_i} \right)^{1-\frac{1}{\sigma}}, & \sigma \neq 1, \\ -\left[\ln \left(\min_i \dfrac{p_i}{1-\beta_i} \right) + 1 \right], & \sigma = 1. \end{cases} \tag{16}$$

Theorem 3 *Assume $\sigma \neq 1$, $\theta > 0$, $k \in \mathbb{R}^n_+$, and either set of assumptions:*

(i) θ satisfies (13);
(ii) $k \in K$, where K is defined by (12), and (11) is satisfied.

Then, the Value Function of the problem of maximizing (15) , subject to (1) is

$$v(k) = \frac{\theta^{-\sigma}}{1-\sigma} \left(\min_i \frac{\eta_i}{1-\beta_i} \right)^{\sigma-1} \langle k, \eta \rangle^{1-\sigma}. \tag{17}$$

Moreover, if

$$N^* = argmin_i \left\{ \frac{\eta_i}{1-\beta_i} \right\}$$

the closed-loop optimal controls are the vectors $c^* \in \mathbb{R}^n_+$ such that

$$c^*_j = 0, \quad \forall j \notin N^* \quad \text{and} \quad \sum_{i \in N^*} c^*_i = \frac{\theta}{\min_i \frac{\eta_i}{1-\beta_i}} \langle k, \eta \rangle. \tag{18}$$

Proof. See Appendix A. □

5. Conclusions

This study delves into a model of distributed capital stocks across nodes, each representing distinct production sites interconnected through a directed, weighted network. The network is used to represent the spillover effects originating from production at one site, impacting the capital stock of neighboring sites.

A primary aspect of the investigation revolves around a regulatory decision to earmark specific nodes for consumption goods production to maximize cumulative utility from consumption. Results highlight the complex relationship between optimal strategies, capital stocks, and both the productivity of resource nodes and the structure of their interconnections. The optimal locations to draw resources for consumption correspond to the nodes with the least eigenvector centrality in a redefined network that merges node productivities with inter-node flows. The study emphasizes the critical role of network structure and node productivity in shaping production, consumption, and resource allocation decisions.

Several open questions remain. The first, and most immediate, is: what happens when the conditions of Theorem 2 are not met and thus the proposed optimal control is not admissible? The theorem's assumptions might not be met for two reasons. The first is that, even if the parameter constraints are met, the system starts from a state outside the cone identified in the statement. How can the dynamics be characterized in this case? How can it be determined if the system still converges to the steady state identified in the statement or not?

The second reason is that the assumptions on the parameters (especially regarding the agent's level of impatience) are not met. What happens in this scenario? What do the optimal controls and trajectories look like? This second question becomes even more relevant in the context of an extension that includes transportation costs, especially when these costs are significant.

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Appendix A. Appendix: Proofs

Proof of Theorem 1. We search for a solution of HJB of type $v(k) = \frac{b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma}$, with $\nabla v(k) = b\langle k, \eta\rangle^{-\sigma}\eta$ so that HJB would imply

$$\frac{\rho b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma} = \frac{\sigma}{1-\sigma}\left(\min_i \frac{\partial v}{\partial k_i}\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(k), [(I + H^\top)\Gamma - D]k\rangle \tag{19}$$

$$= \frac{\sigma}{1-\sigma}\left(\eta_1\right)^{1-\frac{1}{\sigma}} b^{1-\frac{1}{\sigma}}\langle k, \eta\rangle^{1-\sigma} + \lambda b\langle k, \eta\rangle^{1-\sigma} \tag{20}$$

that is

$$b = \left(\frac{\sigma}{\rho - \lambda(1-\sigma)}\right)^{\sigma}\left(\eta_1\right)^{\sigma-1} = \theta^{-\sigma}\left(\eta_1\right)^{\sigma-1}$$

so that

$$c_1^* = b\langle k, \eta\rangle^{-\sigma}(\eta_1)^{-\frac{1}{\sigma}} = \frac{\theta}{\eta_1}\langle k, \eta\rangle, \quad c_i^* = 0,\ i \neq 1.$$

We can then apply a rather standard verification technique (see for instance Fleming and Rishel, 2012), the uniqueness of the optimal control follows by the concavity of the problem (see Acemoglu, 2008). □

Proof of Theorem 2. (i) The only property to check is

$$e^{tA}(K) \subset \mathbb{R}^n_+.$$

By (11), we have that $e^{tA} > 0$ for all $t \geq t_0$, thus $K \subseteq \mathbb{R}^n_+$. Moreover, by definition $k \in K$ implies $k = e^{t_0 A}k_1$ for some $k_1 \in \mathbb{R}^n_+$, which implies that, for every $s \geq 0$,

$$e^{sA}k = e^{(s+t_0)A}k_1 \geq 0.$$

(ii) When instead the stronger assumption (13) hold, it is immediate to check that the matrix A is a Metzler matrix, so that the trajectory $X^*(t) = e^{tA}k$ remains positive at all times, for every initial condition k.

□

Proof of Lemma 1. The validity of (i) is straightforward. For the proof of (ii), we observe first that any generalized eigenvector v corresponding to an eigenvalue $\lambda_i \neq \lambda$ is orthogonal to η. Let's consider an eigenvalue $\lambda_i \neq \lambda$ (which implies $i \geq 2$) and let v_i be an element of the generalized eigenspace V_i. There exists a positive integer m such that $(\Gamma - D + B^\top\Gamma - \lambda_i I)^m v_i = 0$. This leads to:

$$0 = \eta^\top \left[(\Gamma - D + B^\top\Gamma - \lambda_i)^m v_i\right] = \left[\eta^\top(\Gamma - D + B^\top\Gamma - \lambda_i)^m\right] v_i = (\lambda - \lambda_i)^m \eta^\top\, v_i.$$

Since $\lambda \neq \lambda_i$, it follows that $\eta^\top\, v_i = 0$, which means η is orthogonal to v_i. Given the definition of A this fact ensures that v_i is also a generalized eigenvector of A with the eigenvalue λ_i.

Knowing that $\hat{\zeta}$ is an eigenvector for A with the eigenvalue $\lambda - \theta$, it remains to note that the set $\{\hat{\zeta}, v_2, \ldots, v_n\}$ consists of linearly independent vectors. This is evident since $\{v_2, \ldots, v_n\}$ are linearly independent (forming a subset of a basis) and $\hat{\zeta}$ belongs to a distinct generalized eigenspace (of A) from all the V_i for $i \geq 2$, ensuring it cannot be expressed as a linear combination of the v_i for $i \geq 2$. □

Proof of Proposition 1. Since, thanks to Lemma 1 we know that the eigenspace associated to $\lambda - \theta$ has dimension 1 and all other eigenvectors has lower real part than $\lambda-\theta$, it is straightforward to prove that the detrended trajectory converges to some real multiple of $\hat{\zeta}$. So there exists $\alpha>0$ such that

$$\lim_{t\to+\infty} Y(t)=\alpha\hat{\zeta}.$$

On the other hand, from (9), we derive

$$\langle Y(t),\eta\rangle \equiv \langle k,\eta\rangle,$$

Combining these two we derive

$$\langle k,\eta\rangle=\lim_{t\to+\infty}\langle Y(t),\eta\rangle=\langle \lim_{t\to+\infty} Y(t),\eta\rangle=\langle\alpha\hat{\zeta},\eta\rangle,$$

so that

$$\alpha=\frac{\langle k,\eta\rangle}{\langle\hat{\zeta},\eta\rangle}.$$

□

Proof of Lemma 2 . We show first that, as we increase the value of θ, the first component of $\hat{\zeta}$ to become nonpositive is necessarily the first (although other components might also become non-positive simultaneously). By contradiction, assume that $\hat{\zeta}$ is non-negative, with $\hat{\zeta}_1>0$ and $\hat{\zeta}_\ell=0$ for a $\ell\neq 1$. Then, all $\hat{\zeta}_i$ such that $b_{i\ell}>0$, must also be zero. Indeed, if we set $K_\ell=0$, then the equation $\dot{K}_\ell(t)=(\Gamma_i-\delta_i)K_\ell(t)+\sum_{j\neq i} b_{ji}\Gamma_j K_j(t)>0$ arises and this cannot happen (along the eigenvalues of A, the trajectory remains stationary, barring scalar multiplications, thus, if the initial value of K_ℓ is zero, it should stay that way). By iterating this logic for all nodes linked to nodes linked to ℓ and so forth, and given that the graph is strongly connected, all components of the eigenvector must be zero. This contradicts our initial assumption of a strictly positive first component.

So, as we increase the value of θ, the first component to become non-positive is necessarily the first one. When the first component is zero the eigenvector has the form $\hat{\zeta}=(0,\hat{\zeta}_2)$ where $\hat{\zeta}_2$ is a non-negative vector in R^{n-1} and

$$(\lambda-\theta)(0,\hat{\zeta}_2)=A(0,\hat{\zeta}_2)=(a_1,A_{22}\hat{\zeta}_2)$$

so that $a_1=0$ and $\lambda-\theta$ is an eigenvalue of A_{22}. Increasing θ this condition is satisfied the first time when $\hat{\zeta}_2$ is an eigenvector for the dominant eigenvalue λ_{22}. This proves the claim. □

Proof of Proposition 2 . The matrix A has the form

$$A=\begin{pmatrix} a_{11}-\theta & A_{21}-\theta(1,1,...,1)\\ A_{12} & A_{22}\end{pmatrix}$$

so, if we consider an eigenvector v associated to λ_2 we have (restricting our attention to he last $n-1$ components)

$$v_1^\top A_{12} + A_{22}(v_2, .., v_n)^\top = \lambda_2 (v_2, .., v_n)^\top.$$

Now two cases may occur:

(i) if $v_1 = 0$ then $(v_2, .., v_n)$ is an eigenvector of A_{22} but, since λ_{22} is the dominant eigenvalue of A_{22} we have $\lambda_2 < \lambda_{22} < \lambda - \theta$ and we get the claim;

(ii) if $v_1 \neq 0$ we can suppose (up to normalizing the vector) that $v_1 = 1$ and we get

$$A_{12} = (\lambda_2 I - A_{22})(v_2, .., v_n)^\top.$$

We observe that, since v is also an eigenvector for $\Gamma - D + B^\top \Gamma$, which is irreducible, but not the one associated to the dominant eigenvalue, then it necessarily has negative coordinates among $v_2, \dots, v_n$.

If by contradiction, $\lambda_2 > \lambda_{22}$ then $\lambda_2 I - A_{22}$ is invertible and it inverse can be written as

$$\frac{1}{\lambda_2}\left(1 - \frac{A_{22}}{\lambda_2}\right)^{-1} = \frac{1}{\lambda_2}\sum_{k=0}^{\infty}\left(\frac{A_{22}}{\lambda_2}\right)^k$$

so that

$$(v_2, \dots, v_n)^\top = \frac{1}{\lambda_2}\sum_{k=0}^{\infty}\left(\frac{A_{22}}{\lambda_2}\right)^k A_{12}$$

but, since A_{12} is non-negative and each term of the sum $\sum_{k=0}^{\infty}\left(\frac{A_{22}}{\lambda_2}\right)^k$ is non-negative, that would imply that $(v_2, \dots, v_n)^\top$ is non-negative, a contradiction. $\square$

Proof of Theorem 3 . The proof is very similar to that of Theorem 1 so that here we point out only the differences.

We search for a solution of HJB of type $v(k) = \frac{b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma}$, with $\nabla v(k) = b\langle k, \eta\rangle^{-\sigma}\eta$, so that HJB would imply

$$\frac{\rho b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma} = \frac{\sigma}{1-\sigma}\left(\min_i \frac{1}{1-\beta_i}\frac{\partial v}{\partial k_i}\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(k), [(I + H^\top)\Gamma - D]k\rangle \tag{21}$$

$$= \frac{\sigma}{1-\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{1-\frac{1}{\sigma}} b^{1-\frac{1}{\sigma}}\langle k, \eta\rangle^{1-\sigma} + \lambda b\langle k, \eta\rangle^{1-\sigma} \tag{22}$$

that is

$$b = \left(\frac{\sigma}{\rho - \lambda(1-\sigma)}\right)^\sigma \left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{\sigma-1} = \theta^{-\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{\sigma-1}$$

so that

$$q^* = \sum_{i \in N^*} c_i^* = b\langle k, \eta \rangle^{-\sigma} \left(\min_i \frac{\eta_i}{1-\beta_i} \right)^{-\frac{1}{\sigma}} = \frac{\theta}{\min_i \frac{\eta_i}{1-\beta_i}} \langle k, \eta \rangle$$

The remainder of the proof proceeds as in the case of Theorem 1. □

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