ON COMPETITION FOR SPATIALLY DISTRIBUTED RESOURCES IN NETWORKS

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ABSTRACT. Ranking questions regarding settings where a network of flows connects the resource extraction sites have been central in the network literature, however, they have received scant attention in the metapopulation literature. This study examines the dynamics of the exploitation of a natural resource distributed among and flowing between several nodes connected via a weighted, directed network. The network represents the locations and interactions of the resource nodes. A regulator decides to designate some of the nodes as natural reserves where no exploitation is allowed. The remaining nodes are assigned (one-to-one) to players, who exploit the resource at the node. The present study demonstrates how the equilibrium exploitation and resource stocks depend on the productivity of the resource sites, the structure of the connections between the sites, and the number and preferences of the agents. The best locations to host nature reserves are identified per the model's parameters and correspond to the most central (in the sense of eigenvector centrality) nodes of a suitably redefined network that considers the nodes' productivity. The technique proposed in the present study may have applications in decisions regarding the formation of teams when candidate members are heterogeneous in their productivities and connections.
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JEL Classification: Q20, Q28, R11, C73

1. INTRODUCTION

In the exploitation of common property and open access resources, externalities engender distortions social planners or agents may wish to strategically regulate or control. The issue of how to estimate and correct such effects has inspired a huge number of studies in which resource stocks are usually assumed homogeneous in space. Metapopulation models present relevant exceptions (see e.g., Sanchirico and Wilen, 2005) that explicitly address the possibility that natural resource stocks can be *spatially distributed*, with various productive sites connected by non-homogeneous migration flows. Migratory fish provide the most obvious example of a moving distributed stock, but the same spatio-temporal structure is common to other resources, such as water and oil, which are often flowing between locations. Moreover, the same dynamics are shared by other non-natural stocks, such as "knowledge" or pollution, which may be generated in specific locations and, afterward, diffused to others.

In settings where a network of flows connects the resource extraction sites, do different productivities of the various sites and different intensities of migration flow map into a specific hierarchy of the sites? Does this hierarchy affect how the access of competing agents should be regulated and, in particular, where natural reserves should be placed?[1]

While such ranking questions have received scant attention in the metapopulation literature (however, see Costello and Polaski, 2008), they are central in the network literature (for surveys, see Jackson and Zenou, 2015; Zenou, 2016), within which a prominent approach comprises studying the Nash equilibrium of *static* games where players are connected via a network of externalities and identifying the key players in this equilibrium by using network statistics (Ballester et al., 2006).

Accordingly, this study examines the dynamics of the exploitation of a natural resource distributed among and flowing between several nodes connected via a weighted, directed network. It assumes a network perspective on common spatially distributed resources and develops a simple dynamic model where n nodes ($n \geq 2$) of a weighted directed network represent the n sites where the resource resides and evolves in time, while the weights on the edges give the interregional migrations rates of the resource.

[1]The same questions also apply in more general contexts, including mobile resources with environmental or amenity values whose reproduction process is affected by economic activities. For example, given that urban development is likely to worsen conditions at breeding sites of migratory or non-migratory birds that can move, ranking the sites can help inform zoning regulations and urban planning.

The n regions are heterogeneous because they are differently connected, and the growth rates of the resource possibly differ per region. The regulator's task is to assign extraction rights to $f < n$ agents to maximize a welfare function, which, for the most part, we take to be the sum of the agents' utilities. We assume the regulator is constrained to assign at most one agent to a region. Following the assignment stage, the agents compete for the exploitation of the resource as in the classic Levhari and Mirman (1980) dynamic game, with four main differences: 1. Time is continuous, and the exploitation of the resource occurs continuously. 2. The stock of the resource is not homogeneous but distributed among the n regions. 3. The site productivities are independent of the stocks. 4. Each agent can only access the resource through the single node to which they are assigned. Further, we assume the instantaneous utility functions of the agents are isoelastic, consistent with most studies on the Levhari-Mirman game in continuous time. The present study aims to show how the structure of the network affects the regulator's decision.

As the main contribution to the literature, the study shows that when agents are sufficiently "patient" in the generalized growth theory sense that their rate of discount is close to a critical discount rate (see e.g., McFadden, 1973 for a discussion of critical discount rates in optimal growth theory), and the network is (fully or) strongly connected[2], there exists a unique Markov perfect equilibrium (MPE) in linear strategies for the post-assignment dynamic game under two different sets of hypotheses on the structure of agent's action sets (see Theorem 1 and Theorem 2). In this equilibrium, all agents, independently of the assignment, evaluate the different site stocks via a constant common vector of relative prices that proves to be the eigenvector centrality of a network that combines the migration flows and the sites' net rates of growth. These two forces interact in determining the centrality of the sites.

We also provide comparative statics that show how the equilibrium outcome is affected by the choice of sites for natural reserves and the network structure. We begin

[2]Some of this study's results continue to hold or have natural counterparts when source or sink nodes are added to the network, making it reducible. However, we think different kinds of reducibilities yield different phenomena that cannot be captured within a single model.

by showing that when the social planner compares equilibria for different choices of nodes where natural reserves are set, they find that the welfare of each agent who has obtained a permit decreases in the assigned node's centrality measure. Thus, a utilitarian planner always sets the reserves at the most central regions of the network (Section 4.1), and permits are always issued in order of centrality, starting from the most peripheral node.

We then analyze how the outcome changes when the parameters representing the network are varied. In the model, the effects of varying the site productivities and the network density are mediated by the largest eigenvalue of the process that governs the stock's evolution without exploitation. This eigenvalue coincides with the von Neumann rate of growth of the system (i.e., the maximum rate of growth of the resource, or, in dual terms, the rate of interest implicit in the system), and it plays the same role as the productivity parameter in the standard aggregate linear growth model (Rebelo, 1991). The well-known fact that the eigenvalue is an increasing function of the elements of the matrix representing the process can be used to single out the effect of increasing the site productivities but not to study the effect of changing the weights of the network. Indeed, a change in a migration flow engenders a simultaneous change (equal but opposite in sign) in the net growth rate of the node from which the resource flows. Nevertheless, for symmetric networks with different (gross) productivities, we prove that the largest eigenvalue is a decreasing function of the elements of the adjacency matrix.

The present study is naturally related to the metapopulation literature (Hanski, 1999; Sanchirico and Wilen, 2005; Smith et al., 2009; Costello and Polaski, 2008, and others). A few studies in that stream explore aspects of the problem of dynamic strategic interaction with distributed and moving resources, especially to evaluate whether management of the resources through a system of territorial use rights (territorial use right for fishing or "TURF" in the case of fisheries) can effectively mitigate the "tragedy of the commons" (see e.g., Kaffine and Costello, 2011, Costello et al., 2015, Herrera et al., 2016, Costello and Kaffine, 2018, Costello et al., 2019, de Frutos and Martin-Herran, 2019, Fabbri et al., 2020).

Beyond the choice of the time structure (continuous vs discrete), the model we study is similar to the N-patch discrete-time model of Kaffine and Costello (2011). The models, however, differ in the specifications of the production functions (linear vs. strictly concave) and the utility functions (isoelastic vs. linear). The linear specification of the instantaneous utility function significantly simplifies the dynamics in the Kaffine and Costello model, implying that the equilibrium path jumps immediately to the stationary state without any transitional dynamics. Our study's specification, however, allows for rich dynamics.

Among studies recently surveyed by Currarini et al. (2016) on the role of networks in the management of natural resources, İlkılıç (2011) is the closest to the question this study explores. İlkılıç (2011) studies a static game in which a given number of users exploit multiple sources of a common pool, and each user faces marginal costs that are increasing in the total extraction from the site, given the presence of source-specific congestion externalities. The main conclusion is that, in the unique Nash equilibrium of the game, the rate of extraction at each source is proportional to a centrality measure of the links of the source. The model we propose here provides the basis for developing dynamic versions of the İlkılıç (2011) model.

Closely related to the present study is the now extensive literature on differential games in resources economics surveyed in Clemhout and Wan (1994), Dockner et al. (2000), and Long (2011). In almost all the games considered in that literature, the state variable is scalar (an exception is Clemhout and Wan, 1985, where multi-species predator-prey interactions are allowed). Plourde and Yeung (1989) provide a continuous-time version of the Levhari and Mirman (1980) dynamic game. A discussion of the MPE for the case of an exhaustible resource exploited by n agents can be found in Dockner et al. (2000) Section 12.1.2. Clemhout and Wan (1985) provides various models of renewable resources and covers the one-dimensional case in which the reproduction function is linear. The model in this paper is a multidimensional version of the models in Dockner et al. (2000) and Clemhout and Wan (1985). The present study is also broadly related to the network literature that connects the Nash equilibrium of static games to network statistics (e.g., centrality measures) (Ballester et al., 2006, Bramuillé et al., 2014, and Allouch, 2015). We also connect the policy extraction function of the agents in the Markovian equilibrium of the dynamic game to a centrality measure of a network. The eigenvector centrality of the network we study here is also related to the solution of a single-player game and, hence, the Pareto efficient outcomes of the model and the efficiency prices of optimal extraction plans. In the network literature, Elliott and Golub (2019) recently studied a similar problem in a static framework.

The remainder of the paper is organized as follows. Section 2 describes the model and discusses preliminaries. Section 3 presents the main results of the paper and the description of the Nash equilibrium. Section 4 is devoted to comparative statics. Section 5 concludes the study and provides sketches of possible generalizations or applications of the developed techniques to other problems. Appendix A presents the proofs of all analytic results.

2. The Model

We consider a common property resource that is diffused over an area partitioned in subareas or regions. The resource is *mobile* in space, from one region to another, in given proportions. For example, think of fish, mobile across different regional or national waters in seas or oceans. The overall area is modeled here as a network where the nodes represent the different regions, and the weighted edges, the connection intensity. Technically, we consider a directed and weighted network $\mathcal{G}$, with n nodes, as many as the number of regions. The set of nodes is $N := \{1, .., n\}$, and $g_{ij} \geq 0$ is the weight upon the edge connecting a source node i and a target node j, with g_{ij} representing the intensity of the outflow from i to j, so that the $n \times n$ matrix $G = (g_{ij})$, $i, j \in N$, is the adjacency matrix of the migration network $\mathcal{G}$. When

$g_{ij} = 0$ and $g_{ji} = 0$, there are no direct paths between nodes i and j. We assume $g_{ii} = 0$ for all $i \in N$. Moreover, we assume $\mathcal{G}$ is strongly connected; that is, there exists in $\mathcal{G}$ a path connecting any two nodes with corresponding strictly positive coefficients g_{ij}, and $\mathcal{G}$ has no loops. Consequently, the matrix G is irreducible.

The evolution system. We denote by e_i the i-th vector of the canonical basis on $\mathbb{R}^n$, by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$, and by $\mathbb{R}_+$ the set of nonnegative real values. For all $i \in N$, $X_i(t)$ stands for the mass at node i at time t, and we set $X(t) = (X_1(t), \dots, X_n(t))^\top$. The evolution in time of mass $X_i(t)$ on region i depends on several factors:

(a) The natural growth $\Gamma_i X_i(t)$ of the resource at time t at node i, embodied by the (constant) natural growth rate Γ_i; for renewable resources, $\Gamma_i > 0$, and for non-renewable resources, $\Gamma_i \leq 0$;[3]

(b) The outflow of the resource from region i to a linked region j at time t, given by $g_{ij} X_i(t)$, so that the net inflow at location i is given by

$$\left(\sum_{j=1}^{n} g_{ji} X_j(t) \right) - \left(\sum_{j=1}^{n} g_{ij} X_i(t) \right) \quad = \quad \langle Ge_i, X(t) \rangle - \left(\sum_{j=1}^{n} g_{ij} \right) X_i(t).$$

(c) The rates of extraction $c_i(t)$ at time t from region i, which represent the decision variables of the problem and can be chosen by the agents.

Overall, we then have for all i

$$\dot{X}_i(t) = \left(\Gamma_i - \sum_{j=1}^{n} g_{ij} \right) X_i(t) + \langle Ge_i, X(t) \rangle - c_i(t).$$

If $A = (a_{ij})$ is the diagonal matrix of the net reproduction factors, namely $a_{ij} = 0$ if $i \neq j$, and $a_{ii} \equiv a_i = \Gamma_i - \left(\sum_{j=1}^{j=n} g_{ij} \right)$, $c(t) = (c_1(t), .., c_n(t))^\top$, and x_0 is the vector of

[3] If $\Gamma_i = 0$, the resource at node i, unless extracted, remains unchanged in quantity in time (only moving to other nodes). However, $\Gamma_i < 0$ represents resources subject to natural decay.

initial stocks at the different nodes, the evolution of the system in vector form is

$$\begin{cases} \dot{X}(t) = (A + G^\top)X(t) - c(t), \quad t > 0 \\ X(0) = x_0 \in \mathbb{R}^n_+. \end{cases} \tag{1}$$

Moreover, we require the following positivity constraints:

$$c_i(t) \geq 0, \quad t \geq 0, \text{ and } \; X_i(t) \geq 0, \quad \forall\, t \geq 0, \forall i \in N \tag{2}$$

Further, to exemplify what connection weights in G signify, we consider the particular case in which the resource moves toward less crowded areas, proportionally to the difference $X_i(t) - X_j(t)$ (Fick's first law). When such a difference is positive, fish move from node i to node j; when it is negative, from j to i. In this case, $g_{ij} = g_{ji}$, with the net inflow at node i given by

$$-\sum_{j=1}^{n} g_{ij}(X_i - X_j) = \sum_{j=1}^{n} g_{ij}X_j - \sum_{j=1}^{n} g_{ij}X_i.$$

Consequently, $G = G^\top$, then $A + G^\top = A + G$, simplifying the problem.

Harvesting Rules and Payoffs. We assume the regulator uses some of the regions for the reproduction of the resource and assigns the others to agents for exploitation, according to a territorial use right policy. That is, harvesting is prohibited at nodes $i \in M \subset N$, while every node i with $i \in F := N \setminus M$ is assigned exclusively to agent i. Let f be the number of nodes of F, and $n - f$, that of M. We also assume agents interact in a differential game, each maximizing the payoff

$$J_i(c_i) = \int_0^{+\infty} e^{-\rho t} u(c_i(t))dt, \quad i \in F, \tag{3}$$

where $\rho \in \mathbb{R}$ is the discount rate,[4] and

$$u(c) = \ln(c) \quad \textit{or} \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0, \; \sigma \neq 1$$

[4]The results hold regardless of the sign of ρ. Although a negative discount rate is uncommon in applications, a stream of literature considers "upcounting" (see e.g., Le Van and Vailakis, 2005, Dolmas, 1996, and Rebelo, 1991).

(the case of a logarithmic u stands for the case $\sigma = 1$).

2.1. **Primitives of the Network.** We here introduce all relevant parameters of the system and their interpretation. The reader is invited to read this section in parallel with Section 2.2, where the same parameters are computed and commented upon in a toy example. We begin by observing that, without extraction, the elements of matrix $A + G$ capture the joint effects of reproduction and migration; thus $A + G$ can be interpreted as the adjacency matrix of a *signed* network[5] $\mathcal{G}'$, associated with $\mathcal{G}$, where a link from i to j represents how the stock i directly affects stock j, and a loop at i represents how the stock i grows (or decreases) linearly *in situ* (see Figure 1).

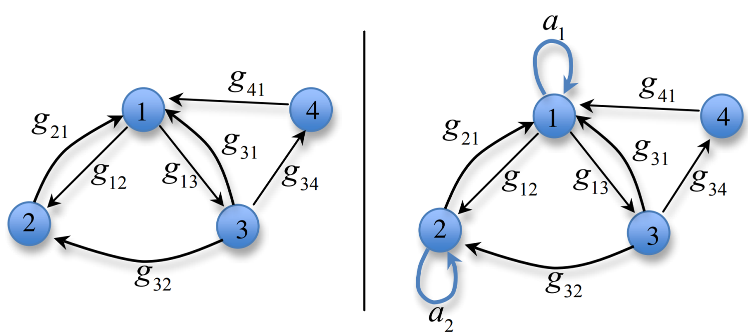

FIGURE 1. A (strongly connected) migration network and the associated signed network, with adjacency matrices G and $A + G$, respectively.

Eigenvalues and eigenvectors of the matrix $A + G$ are essential to this study, as Theorem 1 shows. Note that $A+G$, like G, is irreducible since their elements coincide, except on the diagonal. Moreover, $A+G$ is a Metzler matrix (i.e., it has nonnegative off-diagonal elements). As a consequence of the Perron-Frobenius theorem, $A + G$ has a simple (not necessarily positive) real eigenvalue λ, strictly greater than the real parts of the other eigenvalues and with a *positive* associated normalized eigenvector.[6]

[5]In the literature, *signed network* refers often to networks containing negative links. We borrow the term for the case of nonnegative links but possibly negative loops. As we recall in Footnote 6, negative loops do not preclude the possibility of using the Perron-Frobenius theorem.

[6] The Perron-Frobenius theorem for nonnegative irreducible matrices extends to Metzler irreducible matrices. Indeed, if α is greater than the spectral radius of $A + G$, Perron-Frobenius (see Bapat and Raghavan (1997), Theorem 1.4.4) applies to the nonnegative irreducible matrix $A + G + \alpha I$, with the same eigenvectors as $A + G$'s, whose eigenvalues are $A + G$'s increased by α. In particular, Metzler irreducible matrices have a simple (although not necessarily positive) eigenvalue, associated with a positive eigenvector, strictly greater than the real parts of the other eigenvalues.

The transpose $A + G^\top$ enjoys similar properties. We then order the eigenvalues $\{\lambda, \lambda_2, \lambda_3, \dots, \lambda_n\}$ of $A+G$ as follows:

$$\lambda > \mathrm{Re}(\lambda_2) \geq \mathrm{Re}(\lambda_3) \geq ... \geq \mathrm{Re}(\lambda_n)$$

and call η and ζ respectively the right and left eigenvectors of $A+G$ associated with λ, and both are positive. The rest of the section is then devoted to the interpretation of λ, η, ζ, and associated useful quantities.

2.1.1. *Trajectories in the long run.* The interpretation of the eigenvector ζ is straightforward: ζ represents the long-run direction of convergence of the trajectories of the system when the extraction is null. Indeed, when $c \equiv 0$, the system evolution is entirely ruled by $A + G^\top$, and it is well known that its trajectories converge to the direction of the eigenvector associated with the dominant eigenvalue. Moreover, the trajectories starting on a ray through ζ remain steadily on that ray at all times.

2.1.2. *Weighted Total Mass.* The total mass (or aggregate stock) of the resource is given by $\sum_{i=1}^n X_i(t)$. However, we rather consider the *weighted* total mass[7] given by

$$\langle X(t), \eta\rangle := \sum_{i=1}^n X_i(t)\eta_i.$$

The reason $\langle X(t), \eta\rangle$ is a meaningful way of aggregating stocks is explained in Section 2.1.5. Note that total and weighted total masses grow, in time, at the same rate. Indeed, if $m = \min_i \eta_i$ and $M = \max_i \eta_i$, the fact that $\eta > 0$ implies

$$\frac{1}{M}\langle X(t), \eta\rangle \leq \sum_{i=1}^n X_i(t) \leq \frac{1}{m}\langle X(t), \eta\rangle \leq \frac{M}{m}\sum_{i=1}^n X_i(t), \quad \forall t \geq 0. \tag{4}$$

2.1.3. *Growth Rate of the System.* When $c_i = 0$, $\forall i$ (1) implies $\langle \dot X(t), \eta\rangle = \lambda\langle X(t), \eta\rangle$, and

$$\langle X(t), \eta\rangle = e^{\lambda t}\langle x_0, \eta\rangle.$$

[7]Note that if η is chosen so that $\sum_i \eta_i = 1$, then $\langle X(t), \eta\rangle$ is the weighted average of the $X_i(t)$.

Thus, given (4), the eigenvalue λ represents the *total mass growth rate* without extraction. Moreover, as the expansion in rows of the equality $(A+G)\eta = \lambda\eta$ gives

$$(\lambda - a_i)\eta_i = \sum_{j=1, j\neq i}^{n} g_{ij}\eta_j > 0, \tag{5}$$

with at least one of the g_{ij} strictly positive, the net reproduction rates a_i satisfy

$$a_i < \lambda, \quad \forall i \in F. \tag{6}$$

2.1.4. *Detrended Trajectory.* It is sometimes useful to consider the detrended trajectory of the system without extraction $Y(t) = e^{-\lambda t}X(t)$ so that

$$\dot{Y}(t) = (A + G^\top - \lambda I)Y(t) \tag{7}$$

with null dominant eigenvalue. Hence, $\langle \dot{Y}(t), \eta\rangle = \langle Y(t), (A + G - \lambda I)\eta\rangle = 0$ and

$$\langle Y(t), \eta\rangle \equiv \langle x_0, \eta\rangle, \quad \forall t \geq 0, \tag{8}$$

so that the state $X(t)$ has constant projection in time along the direction of η, magnified by the growth factor $e^{\lambda t}$. Moreover, the motion of the detrended trajectory $Y(t)$ takes place entirely on the simplex

$$\langle y - x_0, \eta\rangle = 0, \quad y \geq 0 \tag{9}$$

obtained by intersecting the plane of $\langle y, \eta\rangle = \langle x_0, \eta\rangle$ with the positive orthant.[8]

2.1.5. *Meaning of the eigenvector η.* Von Neumann prices represent a straightforward interpretation of the components η_i of η; that is, η_i measures the long-term productivity of the system at node i. Indeed, consider the detrended trajectory $Y^i(t)$ starting with a unitary mass concentrated in the i-th node, namely $x_0 = e_i$. Thus, (8) gives $\langle Y^i(t), \eta\rangle = \eta_i$, implying that the total mass, in the long run, is maximized when such unitary mass is allocated in the node where η_i is maximal.

[8]Note that we obtain the same equation as in (9) if we replace x_0 with any other point on the simplex. Moreover, trajectories starting from different points of the simplex are necessarily toward the intersection of ζ with the simplex.

A further interpretation of the η_i is as follows. Assume $\alpha > 0$ is such that the detrended trajectory $Y(t)$ converges to $\alpha\zeta$, as $t \to \infty$. Then (8) implies $\alpha \langle \zeta, \eta \rangle = \langle x_0, \eta \rangle$ and, choosing the norms of η and ζ so that $\langle \zeta, \eta \rangle = 1 = \sum_i \eta_i$, we get

$$\alpha = \sum_i \eta_i x_{0i}, \quad \text{and} \quad \eta_i = \frac{\partial \alpha}{\partial x_{0i}} \tag{10}$$

implying α is a weighted average of the initial stocks x_{0i}, which weights are the η_i, and η_i measures how much site i's initial stock affects the long-run stock $\alpha\zeta$. That is, the trajectory of the system, when the extraction is null, grows toward the long-run direction proportionally to the initial *weighted* total mass $\langle x_0, \eta \rangle$.[9]

Network theory presents another interpretation of η, where η_i represents the *eigen-centralities* of nodes i (not of $\mathcal{G}$, but of the signed network $\mathcal{G}'$). Note that since $(A+G)\eta = \lambda\eta$, and the matrix $\lambda I - A$ is diagonal with all positive diagonal coefficients $\lambda - a_i$, one has

$$(\lambda I - A)^{-1} G \eta = \eta. \tag{11}$$

Then η is the dominant eigenvector (of eigenvalue 1) also of a migration network with adjacency matrix $(\lambda I - A)^{-1}G$; that is, where the coefficients of the original adjacency matrix G are magnified by reproduction rates: the i-th row of G is multiplied by $\frac{1}{\lambda - a_i}$, and flows are magnified by such factor, the greater a_i, the stronger the effect.

Consistently, when the network is complete (i.e., all nodes are positively connected) with equal weights, and the network structure has a neutral effect on nodes, η_i signals the order of net or natural productivity of nodes. Indeed, assume $g_{ij} = \beta > 0$ for all $i \neq j$, and $g_{ii} = 0$. Combining the i-th and ℓ-th row of (5), we get $\eta_\ell = \frac{a_i - \lambda - \beta}{a_\ell - \lambda - \beta}\eta_i$, so that (6) implies $a_\ell - \lambda - \beta < 0$, and hence

$$\eta_\ell \geq \eta_i \quad \Longleftrightarrow \quad a_\ell \geq a_i \quad \Longleftrightarrow \quad \Gamma_\ell \geq \Gamma_i.$$

See Appendix A.1 for further analyses of the hierarchy of nodes in terms of eigen-centrality, through the several examples therein.

[9] The property remains true when agents are active, and the extraction strategies are those described in Theorem 1, with the difference that the net growth rate λ is diminished by the extraction.

2.2. **Toy Examples.** We introduce an example we will work out, through various sections of the paper to illustrate essential concepts. We assume there are three sites (nodes 1, 2, and 3), with the third being the only breeding ground for the resource, namely $\Gamma_1 = \Gamma_2 = 0$, $\Gamma_3 = 2a > 0$. For simplicity, we assume all offsprings leave site 3; thus, the net rate of growth a_3 is zero. The resource flows at rates $a > 0$, $b \geq 0$ across links, so that the migration and the adjacency matrix are, respectively,

$$G = \begin{pmatrix} 0 & b & a \\ b & 0 & a \\ a & a & 0 \end{pmatrix}, \quad A + G = \begin{pmatrix} -(a+b) & b & a \\ b & -(a+b) & a \\ a & a & 0 \end{pmatrix}.$$

Figure 2 illustrates the signed network, with matrix $A+G$. We consider first the evolution of system (1) when $c \equiv 0$. Straightforward calculations yield $\lambda = a$ and $\zeta = \eta = (1,\ 1,\ 2)^\top$ and the non-dominant eigenvalues $\lambda_2 = -2a$ and $\lambda_3 = -(a+2b)$, associated respectively to the left and right eigenvectors (equal, in this case, as the network is symmetric) $(-1,\ -1,\ 1)^\top$ and $(1,\ -1,\ 0)^\top$.

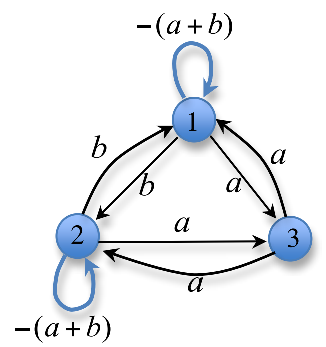

FIGURE 2

Note that site 3 is overall the most central (for it is the most productive) for all values of a, b. It remains true in particular when $a = b$, and the dominant eigenvector of G is $\eta_\circ = (1, 1, 1)$; that is, when the sites have equal centrality in the migration network.

Consistent with (11), another way of analyzing how productivities and the migration network interact is to rewrite equation $(A + G)\eta = \lambda\eta$ as

$$\begin{pmatrix} \frac{1}{2a+b} & 0 & 0 \\ 0 & \frac{1}{2a+b} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} G\eta = \eta.$$

Note that the productivity of a site determines a multiplier of its out-links in the migration network, with higher multipliers for more productive nodes. The same observation also implies the eigenvector centralities of the migration network give the overall centralities when all sites are equally productive.

The detrended trajectory remains on the simplex given by (9) and tends to the intersection between the simplex and ζ (or η), (i.e., to $\alpha\zeta$). Moreover, (7) implies that the vector field $\dot{Y}$ is given by

$$\dot{Y} = \begin{pmatrix} -(2a+b) & b & a \\ b & -(2a+b) & a \\ a & a & -a \end{pmatrix} Y. \tag{12}$$

Now we consider two different instances of this problem. In the first case, we assume $b > 0$, so that the network is complete; in the second case, we assume $b = 0$, so that the network is strongly connected but not complete.

In the case of a *complete network* $(a, b > 0)$, (12) shows that the vector field $\dot{Y}$ is pointing strictly inward at any point Y on the boundary of the simplex (i.e., a Y with one null coordinate), so that the trajectory is entering the interior of the simplex (see Figure 3(a)). This property of straightforward interpretation descends from the completeness of the network: any region where the stock has dropped to zero immediately receives a positive inflow from a region where the stock is positive.

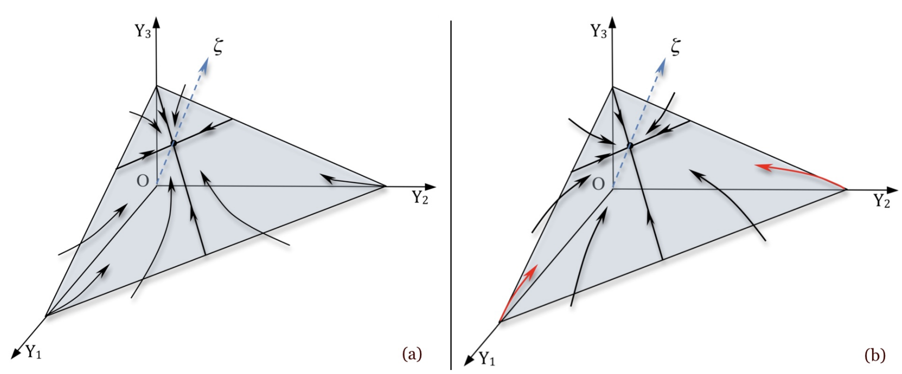

FIGURE 3

By contrast, the property is lost when the migration network is *strongly connected* but not complete, as when $b = 0$. For instance, a trajectory at $Y = (1, 0, 0)^\top$ has associated vector field $\dot{Y} = (-2a, 0, a)^\top$, so that the trajectory is there initially tangent to one side of the simplex, as shown in Figure 3(b). However, the inward-pointing property continues to hold on a subset of the simplex (i.e., in the original 3D space, on a cone contained in the first orthant).

When agents are active with positive extraction, the difference between a complete and a mere strongly connected network has a relevant consequence: in the first case, a small extraction is possible from every initial nonnegative (and non-identically null) value of the stocks, as every node with a potentially null stock receives an instantaneous positive inflow from other nodes; in the second case, the same is true only from stocks in a proper subset of the positive orthant.

3. Existence of Markovian Equilibria

In investigating the Nash equilibria of the game, we restrict the search to stationary Markovian equilibria; that is, when the extraction rates c_i of the agents are described as reaction maps to the observed level of the stock $X(t)$ at time t

$$c(t) = \psi(X(t)), \quad \text{with } c_i(t) = \psi_i(X(t)), \ \forall i \in N$$

(clearly, $\psi_i \equiv 0$, $i \in N \setminus F$). Thus, the system evolves per the *closed-loop equation* (CLE)

$$\begin{cases} \dot{X}(t) = (A + G^\top)X(t) - \psi(X(t)), & t > 0 \\ X(0) = x_0, \end{cases} \tag{13}$$

provided that such an equation has a unique solution. More precisely,

$$\psi = (\psi_1, \psi_2, \dots, \psi_n), \quad \psi_i : S \to [0, +\infty),$$

where S is a subset of $\mathbb{R}^n_+$ (possibly $\mathbb{R}^n_+$ or a cone contained in $\mathbb{R}^n_+$), depending on the data of the problem. We denote the set of admissible strategy profiles by

$$\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

where $\mathbb{A}_i$ is the collection of all (re)actions ψ_i of player i (or a null reaction in nodes with reserves). We denote by $X(t;\psi;x_0)$ or $X^{\psi,x_0}(t)$ the solution of (13). We also adhere to the custom of denoting by ψ_{-i} all components of ψ different from the i-th, so that $\psi = (\psi_i, \psi_{-i})$. As briefly argued at the end of Section 2.2, the choice of S cannot always be the positive orthant $\mathbb{R}^n_+$, differing, for instance, in the cases of a complete or noncomplete network, which justifies the following definition.

Definition 1 **(Consistent couple)** *Assume that, for any $x_0 \in S$ and any $\psi \in \mathbb{A}$, there exists a unique solution X^{ψ,x_0} to* (13), *with $X^{\psi,x_0}(t) \in S$ for all $t \geq 0$. Then, the couple $(S, \mathbb{A})$ is said to be* consistent.

Note that if $(S, \mathbb{A})$ is a consistent couple, the initial stock x lies in S, and players select their strategies in $\mathbb{A}$, then the trajectory always remains in S; that is, S is invariant for strategies in $\mathbb{A}$. This has a direct consequence on the subgame perfection of Nash equilibria, as explained below.

Definition 2 **(Markovian Perfect Equilibrium)** *Assume the couple $(S, \mathbb{A})$ is consistent. We say that a strategy profile $\psi \in \mathbb{A}$ is an* MPE *if, for all $x_0 \in S$ and $i \in F$, the control $c_i(t) = \psi_i(X_i^{\psi,x_0}(t))$ is optimal for the problem of Player i, given by the state equation* (1) *in which $c_j(t) = \psi_j(X(t))$ for every $j \neq i$, the nonnegative state and extraction rates constraints* (2), *and the discounted total payoff $J_i(c_i)$ given by* (3) *to be maximized over the set of admissible controls $\mathbb{A}_i$.*

Hence, if the problem is set at a consistent couple $(S, \mathbb{A})$, a Markovian Nash equilibrium ψ in $\mathbb{A}$ is subgame perfect by definition: if a player deviates (purposefully or mistakenly) from ψ, they cannot leave the set S, and the strategy profile ψ remains feasible for a Nash equilibrium from the state reached in S.

Identifying a suitable consistent couple $(S, \mathbb{A})$ is crucial to ensure subgame perfection. Thus, we must specify the choice of S for different networks (see Section 3.4).

Accordingly, we will proceed by initially assuming that the problem can be set at a consistent couple $(S, \mathbb{A})$, and the equilibrium lies in $\mathbb{A}$ (Theorem 1 in Section 3.1) by later identifying a consistent couple in different sets of data, so that the assumptions of Theorem 1 are satisfied (Section 3.4).

3.1. **Dynamic Programming.** In the forthcoming Theorem 1 and subsequent remarks, we establish the existence of an MPE, compute an explicit formula for the equilibrium, the value function of players, and other relevant quantities. We solve the problem of player i via dynamic programming as follows:

(1) We define V_i, the value function (or *welfare*) of player i as the highest payoff of player i over available choices of the player's strategy c_i, namely

$$V_i(x) = \sup_{c_i \in \mathbb{A}_i} J_i(c_i; x),$$

where x is the initial stock of the resource, and the notation $J_i(c_i; x)$ highlights the dependence in J_i;

(2) Assuming the strategies ψ_{-i} is known, we associate with the problem of player i a Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho v(x) = \max_{c_i \geq 0} \left\{ u(c_i) - c_i \frac{\partial v}{\partial x_i}(x) \right\} + \langle x, (A+G)\nabla v(x) \rangle - \sum_{j \in F - \{i\}} \left(\frac{\partial v}{\partial x_j}(x) \right) \psi_j \quad (14)$$

to which V_i is a (candidate) solution;

(3) We establish the relationship between the maximizing control c_i^* and the value function V_i, $u'(c_i^*) = \frac{\partial V_i}{\partial x_i}(x)$; thus, at every moment, the marginal utility from extraction is equal to the marginal cost of having a smaller amount of the resource in the future at node i. Note that, in both cases of power function and logarithmic utility, u is invertible on the positive real axis such that the previous relationship can be rewritten as

$$c_i^* = \psi_i(x) \equiv (u')^{-1} \left(\frac{\partial V_i}{\partial x_i}(x) \right), \quad (15)$$

which becomes a closed-loop formula for c_i^* once V_i is known;

(4) Finally, in Theorem 1, we exhibit the value functions V_i and a strategy profile $\psi^* = (\psi_i^*, \psi_{-i}^*)$, with linear dependence on the observed stock $X(t)$ that fulfills the above properties for every $i \in F$, thereby yielding an MPE. Specifically, we provide an analytic formula for ψ^* as a function of $X(t)$.

We now set

$$\theta := \frac{\rho + (\sigma - 1)\lambda}{1 + (\sigma - 1)f}. \tag{16}$$

and assume $\theta > 0$.[10]

THEOREM 1 *Assume $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0, \sigma \neq 1$, $\theta > 0$. Assume also that $(S, \mathbb{A})$ is a consistent couple and that $\psi^* \colon S \to \mathbb{R}^n_+$ given by*

$$\psi_i^*(x) = \frac{\theta}{\eta_i} \langle x, \eta \rangle, \text{ for all } i \in F, \qquad \psi_i^*(x) = 0, \text{ for all } i \notin F \tag{17}$$

is a strategy profile in $\mathbb{A}$. Then,

(i) *ψ^* is an MPE of the game in the sense of Definition 2;*

(ii) *the value function of agent i along such equilibrium is*

$$V_i(x) = \frac{\theta^{-\sigma}\eta_i^{\sigma-1}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma}; \tag{18}$$

(iii) *If $X^*(t) = X^{\psi^*, x_0}(t)$ is the trajectory at the equilibrium then*

$$\langle X^*(t), \eta \rangle = e^{\hat{\lambda} t} \langle x_0, \eta \rangle \tag{19}$$

with

$$\hat{\lambda} = \lambda - \theta f = \frac{\lambda - f\rho}{1 + (\sigma - 1) f}. \tag{20}$$

See Appendix A.2 for the proof of Theorem 1.

Several remarks are due here:

[10] Since λ is the implicit rate of growth of the resource, the assumption is the equivalent for distributed resources of the necessary and sufficient condition for the existence of a linear MPE with a homogeneous resource when the f agents have a common constant elasticity of intertemporal substitution and the state equation is linear in the stock (see e.g., Dockner et al. (2000) p. 326, for exhaustible resources). If the condition is violated, the utility of the players is not bounded along the (candidate) equilibrium.

(1) The same result, with due changes, applies to the case of logarithmic utility $u(c) = \ln(c)$. Although we do not discuss this in detail, it can be proven that the associated MPE is obtained from (17) setting $\theta = \rho$, corresponding to the choice $\sigma = 1$ in (16). Consequently, the value function of agent i reads as

$$V_i(x) = \frac{1}{\rho}\left[\ln\left(\frac{\rho}{\eta_i}\langle x, \eta\rangle\right) + \lambda - f\rho\right].$$

(2) For all choices of u, the extraction $\psi_i^*(x)$ and the value function $V_i(x)$ of player i are greater at nodes i with a smaller η_i. Therefore, agents appear to self-regulate, extracting less when at a more central node. However, as in (20), the rate of growth is independent of the agent's assignment; this propensity does not give the regulator a tool to promote the conservation of the resource: an agent optimally extracts less at more central nodes, which offsets the negative effects of exploiting more productive sites.

(3) Note that (18) implies $\frac{\partial V_i(x)}{\partial x_j} = \theta^{-\sigma}\eta_i^{\sigma-1}\langle x, \eta\rangle^{-\sigma}\eta_j$ so that, for any couple of indices j, k in N, one has

$$\frac{\frac{\partial V_i(x)}{\partial x_j}}{\frac{\partial V_i(x)}{\partial x_k}} = \frac{\eta_j}{\eta_k}, \tag{21}$$

where the left-hand side represents the relative shadow prices of the resources at nodes j and k, as evaluated by player i. Nonetheless, the right-hand side does not depend on i, implying that every player gives the same relative evaluation of stocks, independently of the node at which they stand. Section 3.4 further discusses this result.

(4) Equation (19) says that the weighted total mass $\langle X^*(t), \eta\rangle$ of the resource at equilibrium grows with the rate $\hat{\lambda}$, which equals the natural growth rate λ diminished by a quantity proportional to θ and the number of players f. As a consequence of (4), $\hat{\lambda}$ is also the growth rate in the long run of the aggregate stock $\sum_i X_i(t)$.

3.2. **Long-Run Stocks.** We now analyze the long-term behavior of the stock, particularly in establishing if the stock tends to stabilize over time around certain values at different nodes. In section 2.1.1, we noticed that, for a null extraction, the convergence is toward the direction of the eigenvector ζ associated with the dominant eigenvalue λ. Here, we will explain how the equilibrium extraction reduces the growth rate to $\hat{\lambda} = \lambda - \theta f$ and modifies the direction of the associated eigenvector to $\hat{\zeta}$.

We first introduce some useful notation. We define ξ as the vector with components $\xi_i = \eta_i^{-1}$ if $i \in F$, and $\xi_i = 0$ otherwise; and $E := \xi\, \eta^\top$, which is the $n \times n$ matrix obtained by multiplying the column vector ξ by the row vector $\eta^\top$:

$$\xi = \sum_{i \in F} \eta_i^{-1} e_i, \qquad E := \xi\, \eta^\top = (\xi_i \eta_j)_{ij}. \tag{22}$$

We describe the equilibrium trajectory through the eigenvectors/eigenvalues of the matrix of the CLE (13) obtained for $\psi := \psi^*$; that is,

$$\begin{cases} \dot{X}(t) = (A + G^\top - \theta\, E)X(t), & t > 0 \\ X(0) = x_0. \end{cases}, \tag{23}$$

as (22) implies that (17) can be written in vector form as

$$\psi^*(x) = \theta \langle x, \eta \rangle \xi = \theta\, \xi \eta^\top x = \theta E x,$$

(with $\theta = \rho$ in the case of logarithmic utility). We now set

$$\theta_1 = \frac{\lambda - \mathrm{Re}(\lambda_2)}{f},$$

with $0 < \theta < \theta_1$ equivalent to $\hat{\lambda} > \mathrm{Re}(\lambda_2)$. The properties of E (in Appendix A.3, Lemma A.1) imply the following result, proved in Appendix A.3.

Lemma 1 *Assume* $0 < \theta < \theta_1$.

(i) *$A + G - \theta E^\top$ has eigenvector η associated with the eigenvalue $\hat{\lambda} = \lambda - \theta f$; hence, there exists a real eigenvector $\hat{\zeta}$ of $A + G^\top - \theta E$ associated with $\hat{\lambda}$.*

(*ii*) *Consider a basis* $\{\zeta, v_2, \ldots, v_n\}$ *of generalized eigenvectors*[11] *of* $A + G^\top$, *associated with the eigenvalues* $\{\lambda, \lambda_2, \ldots, \lambda_n\}$. *Then* $\{\hat{\zeta}, v_2, \ldots, v_n\}$ *is a basis of generalized eigenvectors for* $A + G^\top - \theta E$ *associated with eigenvalues* $\{\hat{\lambda}, \lambda_2, \ldots, \lambda_n\}$.

The above lemma implies that the extraction process modifies the direction of only the dominant eigenvector of the matrix $A + G^\top$, which changes from ζ to $\hat{\zeta}$, and the associated eigenvalue, which decreases from λ to $\hat{\lambda}$. Notably, the remaining eigenvalues and (left and right) eigenvectors remain the same.

Given that $\hat{\lambda}$ and $\hat{\zeta}$ depend continuously on θ, and $\zeta > 0$, then there exists $\theta_2 > 0$ such that $\hat{\zeta} \equiv \hat{\zeta}(\theta)$ is positive for all $\theta < \theta_2$. We then set

$$\theta_2 = \sup\{\theta : \hat{\zeta}(s) > 0, \forall s \in [0, \theta]\}.$$

The decomposition of the equilibrium trajectory along the eigenvector directions (which the reader finds in Appendix A.3, Lemma A.2) implies the following result.

Proposition 1 *Assume* $0 < \theta < \theta_1$, *let* X^* *be the equilibrium trajectory described in Theorem 1. Then, there exists* $\hat{\alpha} \geq 0$ *such that the detrended trajectory* $X^*(t)e^{-\hat{\lambda}t}$ *satisfies*

$$\lim_{t \to +\infty} X^*(t)e^{-\hat{\lambda}t} = \hat{\alpha}\,\hat{\zeta}.$$

If, in addition, $\theta < \theta_2$, *the trajectory definitively enters the positive orthant.*

See Appendix A.3 for the proof. Notably, the proposition implies the following facts.

First, in the long run, the stock X^* tends to be distributed in the various nodes proportionally to the components of $\hat{\zeta}$. Indeed, when $\hat{\lambda}$ is (remains) the eigenvalue with the greatest real part among $\{\hat{\lambda}, \lambda_2, \ldots, \lambda_n\}$, the equilibrium trajectory X^* converges toward the direction of the associated eigenvector $\hat{\zeta}$.

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[11] A generalized eigenvector of $A + G^\top$ associated with the eigenvalues $\tilde{\lambda}$ is an element of the kernel of $(A + G^\top - \tilde{\lambda}I)^m v = 0$ for some $m \in \mathbb{N}$.

Second, the convergence within the positive orthant is guaranteed by a sufficiently small θ, expressed by the condition $0 < \theta < \min\{\theta_1, \theta_2\}$. That has a straightforward interpretation for logarithmic utility ($\theta = \rho$, corresponding to $\sigma = 1$) given that a small enough θ can be seen as agents being sufficiently patient. If instead $\sigma \neq 1$, and the number of agents is given, $\theta \approx 0$ means $\rho \approx \hat{\rho} \equiv (1-\sigma)\lambda$. In optimal growth theory, $\hat{\rho}$ represents a critical discount rate (the minimum, in the case of one player, for which an optimal solution exists). The case of an exogenous growth rate is dealt with in Brock and Gale (1969), while the case of a linear technology is treated extensively in McFadden (1973)[12]. Given that, with linear technology, there is a trade-off between the growth rate and the intensity of consumption, we can think that agents for whom $\rho \approx \hat{\rho}$ are patient in the generalized sense that they prefer a high growth rate over immediate consumption. The peculiarity in our multiagent setting is that the sign of the difference $\rho - \hat{\rho}$ is not necessarily positive but depends on the sign of the denominator in the formula defining θ.

3.3. **Toy Examples Revisited.** We apply the results in the previous sections to the toy examples of Section 2.2. According to Theorem 1, the equilibrium strategy of a player at nodes 1, 2, or 3 respectively, is

$$\psi_1^*(x) = \psi_2^*(x) = \theta(x_1 + x_2 + 2x_3), \quad \psi_3^*(x) = \frac{\theta}{2}(x_1 + x_2 + 2x_3),$$

with respective value functions

$$V_1(x) = V_2(x) = v^*, \ V_3(x) = \frac{v^*}{2^{1-\sigma}}, \quad \text{where } v^* = \frac{(x_1 + x_2 + 2x_3)^{1-\sigma}}{\theta^\sigma (1-\sigma)}.$$

We now assume there are two agents, assigned to nodes 1 and 2, while node 3 is used as a reserve. Consistent with Lemma 1, the dominant eigenvalue and eigenvector of the matrix $A + G^\top$ (i.e., $\lambda = a$ and $\zeta = (1,1,2)^\top$) are changed by extraction into those of the matrix $A + G^\top - \theta E$

$$\hat{\lambda} = a - 2\theta, \text{ and } \hat{\zeta} = \left(1 - \frac{2\theta}{a}, 1 - \frac{2\theta}{a}, 2\right)^\top,$$

[12]In both studies, time is discrete. However, analogous results hold in continuous time.

while the other (right and left) eigenvalues/eigenvectors remain unchanged. Moreover, $\theta_1 = \frac{3a}{2}$, $\theta_2 = \frac{a}{2}$, and $\hat{\zeta}$ is positive, with the equilibrium trajectory $X^*(t)$ converging toward the direction of $\hat{\zeta}$ whenever $0 < \theta < \min\{\frac{a}{2}, \frac{3a}{2}\} = \frac{a}{2}$.

Finally, to introduce the next section on subgame perfection, we analyze the admissibility of the equilibrium strategy profile at points of $\mathbb{R}^n_+$. In particular, we hint at the fact that the property of the vector field $\dot{Y}$ of pointing strictly inward, highlighted in Section 2.2, plays a role in the admissibility of the strategy profile ψ^*. Equation (19) implies that the detrended trajectory $Y^*(t) = e^{-(a-2\theta)t}X^*(t)$ lies on the simplex $\langle \eta, y - x_0 \rangle = 0$, $y \geq 0$ tending toward the intersection between such plane and $\hat{\zeta}$. The vector field $\dot{Y}^*$ is given by

$$\dot{Y}^* = \begin{pmatrix} \theta - 2a - b & b - \theta & a - 2\theta \\ b - \theta & \theta - 2a - b & a - 2\theta \\ a & a & 2\theta - a \end{pmatrix} Y^*.$$

In the case of a complete network ($b > 0$), ψ^* is admissible at every point of the boundary of the simplex, provided $\theta < \min\{\frac{a}{2}, b\}$, as the vector field remains pointing strictly inward. In the case of a merely strongly connected network ($b = 0$), with similar calculations, one can show that the detrended trajectory $\tilde{Y}$ from, say, $\tilde{Y}_0 = e_1$, starts from a null second coordinate and has $\dot{\tilde{Y}}_2 = -\theta < 0$, thus leaving immediately the positive orthant for every $\theta > 0$. Hence, the equilibrium profile fails to be admissible at some values of $\mathbb{R}^n_+$, let alone subgame perfect. The difference in the two instances of the toy example explains why in the next section admssibility and subgame perfection of the equilibrium are discussed separately for complete and for merely strongly connected networks.

3.4. **Subgame Perfection.** Accordingly, to have subgame perfection of the equilibrium ψ^* in Theorem 1, we must identify a consistent couple $(S, \mathbb{A})$ for the problem: a set of initial data S and a set of strategy profiles $\mathbb{A}$ such that S is invariant for all strategies in $\mathbb{A}$; that is, a stock vector $X(t)$ initially in S remains in S when players choose strategies in $\mathbb{A}$.

3.4.1. *Complete Networks.* We first discuss the case in which the network $\mathcal{G}$ is complete: all nodes are connected by positive edges in both directions: $g_{ij} > 0$, $\forall i, j \in N$ with $i \neq j$, and $g_{ii} = 0, \forall i \in N$. Under this assumption, as large an S as possible can be chosen; that is

$$S = \mathbb{R}^n_+,$$

coupled with a set of strategies $\mathbb{A}_i$ for player i, $i \in F$, given by

$$\mathbb{A}_i := \left\{ \psi_i \colon \mathbb{R}^n_+ \to [0, +\infty) \ : \ \begin{array}{r} (i)\, \psi_i \text{ is Lipschitz-continuous} \\ (ii)\, \psi_i(x) \leq \left\langle (A + G^\top)x, e_i \right\rangle \\ \text{for all } x \in \mathbb{R}^n_+ \text{ such that } x_i = 0. \end{array} \right\} \tag{24}$$

When $i \notin F$, we assume $\mathbb{A}_i$ contains only the null strategy. Note that the Lipschitz-continuity[13] of ψ_i implies that the CLE (13) has a unique solution, $X(t)$, whereas the condition (ii) ensures that, when the stock at node i is null, the extraction ψ_i can be, at most, as much as the inflow at i from the other nodes, so that the stock remains nonnegative at all times. Clearly, $(\mathbb{R}^n_+, \mathbb{A})$ is a consistent couple.

However, is the strategy profile described in (17) in $\mathbb{A}$? The next proposition specifies for which values of the data this situation is true.

Proposition 2 *The strategy profile ψ^* described in* (17) *lies in $\mathbb{A}$ (and consequently, ψ^* is an MPE) if and only if*

$$0 < \theta \leq \hat{\theta} = \min \left\{ g_{ij} \frac{\eta_i}{\eta_j} \ : \ i \in F, \, j \in N, \, i \neq j \right\}. \tag{25}$$

Specifically, if (25) *is violated, there exist initial data $x_0 \in \mathbb{R}^n_+$ such that the trajectory $X^*(\cdot)$ starting at x_0 leaves the positive orthant $\mathbb{R}^n_+$ at some times.*

The proof is in Appendix A.4. Note that $\hat{\theta} > 0$, given the full connection of the network. However, Condition (25), again, requires that agents are sufficiently patient in extracting the resource.

[13]A ψ_i is Lipschitz-continuous if there exists $L > 0$ such that $|\psi_i(x) - \psi_i(y)| \leq L|x - y|$, for every $x, y \in \mathbb{R}^n_+$. Lipschitz-continuous functions are also differentiable a.a. (with respect to Lebesgue measure) with a bounded derivative, at the points where it exists.

3.4.2. *Strongly Connected Networks.* For a strongly connected network where some $g_{ij},\ i \neq j$ are null, Proposition 2 implies that there exist some initial positions x_0 from which the trajectory X^* solving (23) is not feasible; that is, $X^*(t)$ leaves the positive orthant, at least for some time t. Nonetheless, exhibiting a consistent couple is possible, at least with *capacity constraints*, expressing that extraction is more challenging when the resource is less abundant. More precisely, we assume

$$\psi_i(x) \leq \beta_i x_i, \quad \forall i \in F$$

where $\beta_i \geq 0$ are given catchability parameters, and define $\mathbb{A}^\beta = \mathbb{A}_1^{\beta_1} \times \cdots \times \mathbb{A}_n^{\beta_n}$, where the strategy set of player i is

$$\mathbb{A}_i^{\beta_i} := \left\{ \psi_i \colon S \to [0, +\infty) \ : \ \begin{array}{l} (i)\, \psi_i \text{ is Lipschitz-continuous} \\ (ii)\, \psi_i(x) \leq \beta_i x_i, \text{ for all } x \in S. \end{array} \right\}$$

Strategies in $\mathbb{A}_j^{\beta_j}$ at reserve nodes $j \notin F$ are chosen null.

We now establish the existence of a cone S in $\mathbb{R}_+^n$ such that $(S, \mathbb{A}^\beta)$ is a consistent couple. For a positively defined matrix $P \in \mathbb{R}^n \times \mathbb{R}^n$ and a real constant d, we define the ellipsoid depending on P and d

$$E(P,d) := \left\{ x \in \mathbb{R}_+^n \ : \ x^\top P x \leq d \right\},$$

and the cone through the origin containing the positive eigenvector ζ

$$S^* = \{ry : r > 0, y \in \zeta + E(P,d)\}. \tag{26}$$

Note that for a small enough d, $E(P,d)$ is an arbitrarily small neighborhood of the origin, regardless of the choice of P. Consequently, for a small enough d, both $\zeta + E(P,d)$ and S^* are entirely contained in $(0,+\infty)^n$.

PROPOSITION 3 *Let ψ^* be the strategy profile described in* (17)*, $X^*(t) := X(t; x_0, \psi^*)$, the associated trajectory, and S^*, the cone defined in* (26) *and contained in* $(0,+\infty)^n$*. For small enough catchability parameters $\beta_i > 0$, there exists a threshold*

$\theta_\beta \in (0, \min\{\theta_1, \theta_2\})$ *such that when* $\theta \in (0, \theta_\beta)$ *the couple* $(S^*, \mathbb{A}^\beta)$ *is consistent, the strategy profile* ψ^* *is admissible, and, hence,* ψ^* *is an MPE in* $(S^*, \mathbb{A}^\beta)$.

See Appendix A.4 for the proof.

3.5. **Uniqueness of the Equilibrium.** Finding all the MPE for the problem would require simultaneously solving f interdependent partial differential equations of type (14), one for every player. When the state variable is scalar, the system reduces to a system of ordinary differential equations, for which standard uniqueness results can sometimes be used (e.g., Cvitanić and Georgiadis, 2016), but with a state variable dimension of at least two, we are not aware of any uniqueness result for systems of PDE of such general form. Nonetheless, assuming that linear strategies are salient because of their simplicity, and players are, therefore, more likely to coordinate on this kind of equilibria rather than (possibly existing) alternative equilibria with a more complex structure, we analyze uniqueness among linear strategies.

Regarding the strategy profile ψ^* given by (17), we can prove that ψ^* is the unique linear MPE for the problem:

(*i*) on $(\mathbb{R}^n_+, \mathbb{A})$, when the network $\mathcal{G}$ is complete;
(*ii*) on $(S^*, \mathbb{A}^\beta)$, for small enough $\beta > 0$ and $\theta > 0$, when $\mathcal{G}$ is strongly connected.

Theorem 2 *Assume* $\mathcal{G}$ *complete,* $\theta \in (0, \hat{\theta})$*, and* $f < n$*. The hypotheses of Theorem 1 are then verified at the consistent couple* $(\mathbb{R}^n_+, \mathbb{A})$*, and the strategy profile* ψ^* *described is the unique linear MPE of the game.*

See Appendix A.5 for the proof of this theorem, though we explain the main ideas by working out the example with a complete network in Section 2.2, where we set $b = a > 0$. Assume, in equilibrium, two players harvesting at nodes 1 and 2 using the linear strategies $c_j(t) = \langle w^j, X(t)\rangle \equiv w^j_1 X_1(t) + w^j_2 X_2(t) + w^j_3 X_3(t)$, with $w^j_i \geq 0$, while node 3 is the reserve. The evolution of the system is then $\dot{X}(t) = (A + G^\top - W)X(t)$, with the matrix W having rows w^1, w^2, and $w^3 = 0$. More explicitly,

$$(A + G^\top - W) = \begin{pmatrix} -2a - w_1^1 & a - w_2^1 & a - w_3^1 \\ a - w_1^2 & -2a - w_2^2 & a - w_3^2 \\ a & a & 0 \end{pmatrix}.$$

We show that this linear equilibrium necessarily coincides with (17).

We denote by W_{-j} the matrix W where the j-th line is replaced by a vector of zeros, so that $W = W_{-1} + W_{-2}$. For $w_i^j < a$ for all $i \neq j$, both $A + G^\top - W$ and $A + G^\top - W_{-j}$ are Metzler matrices with strictly positive out-of-diagonal entries. Thus, as in Footnote 6, they have a single dominant eigenvalue, respectively ℓ_W and ℓ_j, with positive left eigenvectors η^W and η^j. At the equilibrium, each agent j maximizes the utility, given the evolution described by $\dot{X}(t) = (A + G^\top - W_{-j})X(t) - e_j c_j(t)$. A standard optimization argument applied separately to both players shows that their optimal policy is linear; more precisely, $c_j(t) = d_j \langle X(t), \eta^j \rangle$, with $d_j = \theta_j \langle \eta^j, e_j \rangle^{-1}$ and $\theta_j = (\rho + (\sigma - 1)\ell_j)\sigma^{-1}$. Consequently, $W_{-1}^\top = d_2 \eta^2 e_2^\top$, and $W_{-2}^\top = d_1 \eta^1 e_1^\top$. From

$$\ell_1 \eta^1 = (A + G - W_{-1}^\top)\eta^1 = (A + G - W^\top)\eta^1 + W_{-2}^\top \eta^1 = (A + G - W^\top)\eta_1 + \theta_1 \eta^1,$$

we derive $(A + G - W^\top)\eta^1 = (\ell_1 - \theta_1)\eta^1 = \frac{\ell_1 - \rho}{\sigma}\eta^1$. Since by Perron-Frobenius the only positive eigenvector of $A + G - W^\top$ is η^W (except for positive multiples), one has $\eta^1 = \eta^W$ and, similarly, $\eta^2 = \eta^W$. Hence, in any linear MPE, all policy functions have the form $c_j(t) = d_j \left\langle X(t), \eta^W \right\rangle$, so that *all agents tacitly agree on the relative values of the stocks in the different sites*. Moreover, given that

$$\begin{aligned}(A + G)\eta^W &= (A + G - W)\eta^W + W\eta^W = (A + G - W)\eta^W + W_{-1}^\top \eta^W + W_{-2}^\top \eta^W \\ &= (A + G - W)\eta^W + W_{-1}^\top \eta^2 + W_{-2}^\top \eta^1 = (\ell^W + (\theta_1 + \theta_2))\eta^W,\end{aligned}$$

η^W is the unique positive eigenvector of $(A + G)$; hence, $\eta^W = \eta$. Thus, *the above consensus can only be reached in terms of the eigenvector centralities of the original network* $A + G$. Finally, with further calculations, one proves that the coefficients d_j are necessarily those described in Theorem 1.

The presence of reserves is quite primal in the study subject; thus, the assumption $f < n$ in Theorem 2 can be considered not to be particularly strong. However, note that this assumption is indispensable: one can construct examples of systems with $f = n$ (which verifies the other assumptions of the theorem) in which more than one linear equilibrium can be constructed. However, these examples are specific, and multiplicity of linear equilibria likely occurs only on a subset of measure zero of the space of admissible parameters.

A counterpart of the result proved in Theorem 2 can be stated for the second consistent couple characterized in Proposition 3; that is, the cone and the effort-constrained controls. Here, we do not need to restrict to complete networks but must work with sufficiently small catchability parameters β_i. The result is as follows.

Theorem 3 *Suppose the hypotheses of Proposition 3 are verified. If* $\theta \in (0, \theta_1)$ *and* $\beta > 0$ *are small enough, the MPE given in Theorem 1 and Proposition 3 is the unique linear MPE of the game on* $(S^*, \mathbb{A}^\beta)$.

Appendix A.5 presents a sketch of the proof (similar to and even simpler than the one of Theorem 2).

4. Comparative Statics

4.1. **Optimal Location of the Reserves.** Here, we assume the number f of extraction permits is given, and intervention of the regulator is limited to deciding where the $n - f$ natural reserves are placed among the available n regions. This decision is made at the beginning of the game and never changed afterward.

We assume agents choose their strategies according to Theorem 1, and such equilibrium strategies are admissible for the given set of data for any choice of reserves placement. The regulator then compares the outcome associated with the different placements and chooses the configuration maximizing the sum of utilities of players. Hence, if $\mathcal{F} = \{F \subset N : |F| = f\}$ describes all possible subsets of N having f

elements, the regulator maximizes with respect to $F \in \mathcal{F}$

$$W(x, F) = \sum_{i \in F} V_i(x), \tag{27}$$

where value functions V_i are those described in Theorem 1, and x is the initial distribution of the resource through the nodes.

PROPOSITION 4 *Under the assumptions of Theorem 1, assume the strategies profile ψ^* is admissible at x_0 for any choice of $F \in \mathcal{F}$. The social welfare W defined in* (27) *is then maximized if the natural reserves are built at a subset F of $n-f$ nodes i, where η_i are highest. If $F^* \in \mathcal{F}$ is one of such choices, then*

$$\max_{F \in \mathcal{F}} W(x_0, F) = W(x_0, F^*) = \frac{\theta^{-\sigma}}{1-\sigma} \langle x_0, \eta \rangle^{1-\sigma} \sum_{i \in F^*} \eta_i^{\sigma-1}. \tag{28}$$

The proof is straightforward, as each term of the sum in (28) is a nonincreasing function of η_i. This result has a clear explanation. In Section 2.1, we show that η_i measures the long-term productivity at node i, and, in (21), every player gives the same evaluation of stocks, independent of the node at which they stand. Moreover, as observed in Section 3.2, “patient” agents (i.e., agents whose optimal rate of extraction θ is small) prefer to consume dividends rather than stocks. Given that resource flows extracted in the different regions are perfect substitutes, the regulator must optimally preserve those units of stocks that prospectively have higher productivity.

In the examples of Section 2.2 with two players, where the dominant eigenvector is $\eta = (1, 1, 2)$, the social planner optimally places the reserve at node 3. Moreover, in the examples in Appendix A.1, the more central nodes (in terms of eigencentrality) are the optimal choice for reserve placement.

4.2. **Comparative Growth Rates.** We now analyze how the long-term growth rate of the resource stocks (i.e., $\hat{\lambda}$ given by (20)) changes with respect to the parameters of the system, such as reproduction rates Γ_i, embodying local productivity advancements, the largest eigenvalue λ of $A+G$ representing the maximum rate of growth of the system (growth will null extraction), and the number of players f.

We assume, as in Theorem 1, that the equilibrium extraction rate θ defined in (16) is strictly positive, with the condition satisfied with a positive numerator and denominator in (16). We refer to this case as the *standard regime*,as opposed to the case of a negative numerator and denominator, which is the *voracious regime* briefly discussed at the end of this section.[14] As easily computed, the standard regime occurs under mutually exclusive conditions (we consider "standard" also the case of logarithmic utility, corresponding to $\sigma = 1$):

(a) $f \geq 1,\ \sigma > 1,\ \lambda > -\dfrac{\rho}{\sigma - 1}$;
(b) $1 \leq f < \frac{1}{1-\sigma},\quad 0 < \sigma < 1,\quad \lambda < \dfrac{\rho}{1-\sigma}$,;
(c) $f \geq 1,\ \sigma = 1$.

Appendix A.6 presents the proof of the following proposition.

Proposition 5 *Under the assumptions of Theorem 1 and in the standard regime, represented by any of the subcases* (a), (b), *and* (c), *consider the growth rate* $\hat{\lambda}$ *of the system described by* (20). *Then,*

(*i*) $\hat{\lambda}$ *is strictly increasing in* Γ_i, *for all* i;
(*ii*) $\hat{\lambda}$ *is strictly increasing in* λ.
(*iii*) $\hat{\lambda}$ *is strictly decreasing in* f.

The picture emerging from (*i*) and (*ii*) in Proposition 5 is transparent and justifies denoting the said regime as "standard." Moreover, (*iii*) confirms that a tragedy of commons mechanism prevails: the higher the number of agents the quicker they tend to appropriate the resource to avoid being preceded by the others. We further add the following:

[14]The term "voracious" originates from Tornell and Lane (1999) in the context of a development model with a single common stock and multiple private stocks. The two regimes recur and yield significantly different system behaviors.

- When $f = 1$, a positive numerator $\rho - (1-\sigma)\lambda$ in (16) is a necessary and sufficient condition for a finite value function. Moreover, as λ represents the asymptotic growth rate of the resource under null extraction, the result is consistent with the parallel condition $\rho - (1-\sigma)A > 0$ in the standard single-player AK-models (for extraction or growth). The same remark applies to the problem of a social planner maximizing the sum of utilities of f players (see e.g., Freni et al., 2006).
- Now, we consider a game with f players. In subcase (b), the condition $0 < \sigma < 1$ coexists with a restriction on the number of players, $f < \frac{1}{1-\sigma}$. The latter descends from the fact that each agent, in solving their control problem, perceives a maximum growth rate of the resource of $\lambda - (f-1)\theta$. Hence, their value function is finite if and only if $\rho - (1-\sigma)[\lambda - (f-1)\theta] > 0$; that is, $1 + (\sigma - 1)f > 0$. Similar conditions for the aggregate cases with $A = 0$ are given in Dockner et al. (2000).
- Referring to subcases (a), (b), and (c) of the standard regime, note that

$$\frac{d\theta}{d\lambda} = -\frac{1-\sigma}{1-(1-\sigma)f} = \frac{1}{f-(1-\sigma)^{-1}}.$$

  Hence, in case (a), $\frac{d\theta}{d\lambda} > 0$, meaning that the agents exploit the resource more intensively when λ is greater; in case (b) $\frac{d\theta}{d\lambda} < 0$, with an opposite interpretation; in case (c) $\theta = \rho$ (corresponding to $\frac{d\theta}{d\lambda} = 0$), the extraction is independent of λ. Interpreting λ as the rate of interest of the system, this fact has a natural explanation, recalling that with a unitary intertemporal elasticity of substitution (the case of a logarithmic utility) the substitution and the income effects associated with a change in the rate of interest exactly cancel each other out.

The growth rate $\hat{\lambda}$ also changes with changes in the entries of the adjacency matrix G, although less predictably. Indeed, we cannot use the fact that λ is an increasing function of the elements of the matrix $A + G$, as a change in a migration flow G engenders a simultaneous change (equal but opposite in sign) in the net growth rates constituting A. In the case of symmetric networks, we can establish the following result (proof in Appendix A.6).

Proposition 6 *Assume G is symmetric (i.e. $g_{ij} = g_{ji}$, for all $i, j \in N$) and that assumptions of Theorem 1 hold. Then, for all $i, j \in N$*

(i) *λ is a nonincreasing function of g_{ij}, strictly decreasing if $\Gamma_i \neq \Gamma_j$;*
(ii) *$\hat{\lambda}$ is a nonincreasing function of g_{ij}.*

Thus, we interpret that increased mobility prevents the accumulation of the stock in more productive sites, causing a decrease in the maximum growth rate λ.

4.2.1. *The voracious regime.* In the voracious regime, the equilibrium extraction rate θ defined by (16) bears a negative numerator and denominator; that is[15]

$$\rho - (1-\sigma)\lambda < 0, \; 0 < \sigma < 1, \; f > \frac{1}{1-\sigma}. \tag{29}$$

As a consequence, we have

$$\frac{d\theta}{d\lambda} > \frac{1}{f} > 0, \quad \text{and} \quad \frac{d\hat{\lambda}}{d\lambda} = \frac{1}{1-(1-\sigma)f} < 0,$$

the former establishing that the agents exploit the resource more intensively when λ is greater and the latter establishing that the long-term growth rate decreases although the maximal growth rate of the system increases, implying that the associated increase in the extraction rate θ is somewhat disproportionate, hence, "voracious." Moreover, in settings of parameters satisfying (29), the conclusions of Propositions 5 and 6 reverse: $\hat{\lambda}$ is decreasing in λ, increasing in f in Γ_i, and increasing in g_{ij} when G is symmetric.[16]

[15]Some studies report an elasticity of intertemporal substitution $\frac{1}{\sigma}$ close to zero (see e.g., Hall, 1988 and Best et al., 2020), while others report values greater than 1 (e.g., Gruber, 2013).

[16]In the voracious regime (29), the social planner problem is not well defined, as some strategies engender an infinite value function. However, if there is a sufficiently high number of agents, the game has an MPE, as the players, by overexploiting the resource, reduce future earnings and, thus, keep their value functions finite.

5. Conclusion

By using a simple framework with heterogeneous regions and a given number of agents the present study aimed to ascertain how the structure of the migration network and the sites productivities affect competition for spatially distributed moving resources. We found that if the regulator's objective is to maximize the unweighted sum of the utilities of the agents, and they are constrained to assign no more than one agent to each region, the reserves should be localized in the most central regions, with the relevant centrality measure being given by the eigenvector centrality of a network whose links are the coefficients of the migration flows and whose loops are the *in situ* net productivities of the nodes.

Although the agents and the regulator in the analysis care only about resource consumption, the model provides a basis for more general analyses in which preferences for conservation are considered. For example, resource stocks could be introduced in the utility functions of the agents or the regulator welfare function. It is plausible that, under the new assumptions, a strong bias toward stock conservation could even induce a reversal of the above assignment rule. Moreover, the role of the regulator could additionally be examined in more general contexts in which a "bad" extreme equilibrium coexists with the interior equilibrium. For example, an extreme equilibrium can be expected to exist in variants of the model in which the extracted resource can be stored (e.g., Kremer and Morcom, 2000). In this case, a spatially structured policy could be a useful tool to eliminate the incentives that might potentially induce agents to coordinate on the "bad" outcome.

We derived the results in the context of a game of extraction of a natural resource. However, there are other interpretations of the model; thus, it can find applications in games of a different nature, such as *growth models with externalities*, *dynamic games with spatial diffusion of pollution*, and *dynamic contribution games*. For example, from the analysis, one can study a growth model where production is distributed among n (differently productive) nodes and the production on one node generates nonnegative externalities, captured by the matrix G, on the production of the other nodes. In this context, extensions of the Perron-Frobenius theory to matrices with some negative entries (e.g., eventually positive or eventually exponentially positive matrices; see e.g., Noutnos and Tsatsomeros, 2008) could be used to further extend the analysis to cases in which positive and negative externalities coexist. Alternatively, by interpreting the state variables as local measures of environmental quality and the weighted sum $\langle X(t), \eta \rangle$ as the corresponding aggregate measure, the model can be applied to study a dynamic pollution game in which "clearness" moves across different locations and is depleted by the agents' local economic activities. Direct costs of pollution can be considered by assuming stock-dependent utility functions.

Finally, the model could be adapted to study a discrete public goods contribution game where a group of agents invests in, for example, knowledge to reach a target, given externalities (e.g., imagine multiple connected laboratories trying to achieve a scientific breakthrough). In that case, the control variables must be interpreted as costly efforts that influence the state of the project. Homogeneous-stock versions of the model have been examined by scholars such as Kessing (2007), who show that efforts are strategic complements in time; Georgiadis (2015), who analyze optimal contracts for a generalized model in which the evolution of the project is stochastic; and Cvitanić and Georgiadis (2016), who propose a budget-balanced mechanism that induces each agent to choose the first-best effort level. An adaptation of the technique we use can be utilized to generalize the above mentioned one-dimensional dynamic contribution games and may have applications in the decisions regarding the formation of teams when candidate members are heterogeneous in their productivities and connections.

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Appendix A. Proofs and Complements

A.1. **More on Nodes Centrality.** We here discuss the hierarchy of nodes in some more examples.

Example 1. We consider a network in which net productivities are all equal

$$a_i = \Gamma_i - \sum_{j=1}^{n} g_{ij} \equiv a \quad \text{for all } i \in N.$$

We denote the Perron–Frobenius eigenvalue for G by $\lambda_\circ$ and the associated normalized eigenvector by $\eta_\circ$. In this context, $A + G = aI + G$, and the eigenvectors of G and $aI + G$ are the same, implying $\eta = \eta_\circ$ (with η, $\eta_\circ$ is associated, respectively, with eigenvalues λ, and $\lambda_\circ = \lambda - a$). Hence, when nodes are equally productive, all sites are ranked according to the eigenvector centrality $\eta_\circ$ of the migration network $\mathcal{G}$, with η_i higher when node i is better connected to the other nodes (i.e., more eigencentral). That is, when nodes are undifferentiated with respect to productivity, the migration network rules the hierarchy.

Example 2. The last example in Section 2.1.5 showed that when the network is complete with equal weights, then $\eta_\ell \geq \eta_i \iff a_\ell \geq a_i \iff \Gamma_\ell \geq \Gamma_i$. We wonder if the property is true in general. Specifically, if a node ℓ has a greater centrality than a node i in the migration network $\eta^\circ_\ell \geq \eta^\circ_i$, and a greater reproduction rate $\Gamma_\ell \geq \Gamma_i$, is it true that the reserve is better placed at node ℓ than at node i (i.e., $\eta_\ell \geq \eta_i$)? The answer is negative, as explained in the following example. Consider the network described by $\Gamma_1 = 1, \quad \Gamma_2 = 1 + b, \quad \Gamma_3 = 0,$

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad A + G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b & 1 \\ 2 & 0 & -2 \end{pmatrix},$$

with $b > 0$. An explicit calculation yields $\lambda_\circ = \sqrt[3]{2}$ and $\eta^\circ = (1, \sqrt[3]{2}, \sqrt[3]{4})^\top$. Note that $\eta^\circ_2 > \eta^\circ_1$ and $\Gamma_2 > \Gamma_1$; that is, node 2 precedes node 1 both in productivity (natural and net) and centrality. Nonetheless, $\eta_1 > \eta_2$ for some choices of positive b, as we show next. If $\eta = (1, \eta_2, \eta_3)^\top$, from $(A + G)\eta = \lambda\eta$, we derive

$$\eta_1 = 1, \quad \eta_2 = \lambda, \quad \eta_3 = \lambda(\lambda - b), \quad b = \lambda - \frac{2}{\lambda(\lambda + 2)}.$$

Note that the last equation implies that b is an increasing function of λ and vice versa. A direct calculation shows that for $b = 0$, one has $\lambda(0) \simeq 0.8$, so that by continuity $\lambda(0) < \lambda(b) < 1$ for small positive b. Hence $\eta_1 > \eta_2$ and a reserve is better set at node 1 rather than at node 2. Thus, the relationship between the hierarchy dictated by the eigencentrality η and the productivity/network structure is complex and generally nonmonotonic.

Example 3. Finally, we interpret η_i as a measure of productivity and connectiveness of the i-th node and the nodes more directly connected to it. We take G, as in the previous example, and set $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = 2 + a$. Then

$$\mu_1 = 1, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda^2, \quad a = \lambda - \frac{2}{\lambda^2},$$

with λ an increasing function of a; moreover, for $a = -1$ one has $\mu_1 = \mu_2 = \mu_3 = 1$, and $\lambda = 1$, so that $\lambda > 1$ if and only if $a > -1$. Therefore,

$$\mu_1 < \mu_2 < \mu_3 \text{ for } a > -1, \quad \text{and} \quad \mu_1 > \mu_2 > \mu_3 \text{ for } a < -1.$$

Hence, an increasing reproduction rate Γ_3 increases η_3, making (definitively) node 3 the most central, and influences the centrality η_2 of node 2, which is more directly connected to it than node 1.

A.2. **Proofs for Subsection 3.1.**

Proof of Theorem 1. We initially take the perspective of player i, active at node i. For all other players, we assume that they play the strategies

$$\psi_j(x) = \frac{\theta}{\eta_j} \langle x, \eta \rangle, \text{ with } j \in F \setminus \{i\}.$$

With this choice, the HJB equation (14) becomes

$$\rho v(x) = \frac{\sigma}{1-\sigma} \left(\frac{\partial v}{\partial x_i} \right)^{1-\frac{1}{\sigma}} + \langle x, (A+G)\nabla v(x) \rangle - \langle x, \eta \rangle \sum_{j \in F \setminus \{i\}} \left(\frac{\partial v}{\partial x_j} \right) \frac{\theta}{\eta_j},$$

with the maximum attained at

$$c_i = \left(\frac{\partial v}{\partial x_j}\right)^{-\frac{1}{\sigma}}. \tag{30}$$

Step 1: we search for a solution of the HJB equation of type

$$v(x) = \frac{b_i}{1-\sigma} \langle x, \eta \rangle^{1-\sigma}, \quad \text{with} \quad \nabla v(x) = b_i \langle x, \eta \rangle^{-\sigma} \eta, \tag{31}$$

where b_i is a suitable positive real number. Substituting v and its partial derivatives into the HJB equation yields v as a solution if and only if

$$b_i = \frac{1}{\eta_i} \left(\frac{\sigma \eta_i}{\rho - \lambda (1-\sigma) + (1-\sigma)(f-1)\theta} \right)^{\sigma}.$$

Given that from (16), $\rho - \lambda (1-\sigma) = \theta(1 + (\sigma - 1) f)$, last equation gives

$$b_i = \frac{1}{\eta_i} \left(\frac{\eta_i}{\theta} \right)^{\sigma}.$$

Step 2: Markovian equilibrium. From (30) follows

$$\psi_i(x) = (b_i \eta_i)^{-\frac{1}{\sigma}} \langle x, \eta \rangle = \frac{\theta}{\eta_i} \langle x, \eta \rangle,$$

so that the candidate equilibrium is given by (17), and the associated value functions, (18). We prove in Step 4 that (17) is indeed an equilibrium.

Step 3: Closed-loop equation. Note that $c(t) = \theta \langle X(t), \eta \rangle \xi = \theta \, \xi \eta^\top X(t)$ along the equilibrium trajectories, implying the evolution system can be rephrased as in (23). Statement *(iii)* follows from

$$\langle \dot{X}(t), \eta \rangle = \langle X(t), (A+G)\eta \rangle - \langle X(t), \eta \rangle \langle \xi, \eta \rangle = \langle X(t), \eta \rangle (\lambda - \theta f),$$

where $\lambda - \theta f = \hat{\lambda} = (\lambda - f\rho)(1 + (\sigma - 1) f)^{-1}$.

Step 4: Best response. We verify now that the feedback strategy (17) is the best response for Player i, when the other players choose ψ_j, with $j \neq i$, as in (17). The problem of Player i is then maximizing (3) under the dynamics

$$\begin{cases} \dot{X}(t) = (A + G^\top - \theta\, \xi^i \eta^\top)X(t) - c_i(t)e_i, & t > 0 \\ X(0) = x_0. \end{cases}, \tag{32}$$

where $\xi^i_\ell = \xi_\ell$ for all $\ell \neq i$, and $\xi^i_i = 0$. Set $c_i^*(t) = \psi(X^*(t))$, and let $c_i(t)$ be any other admissible control, with $X^*(t)$ and $X(t)$, respectively, as the associated trajectories. Now we consider the quantity $(c_i^*(t) - c_i(t))\frac{\partial v}{\partial x_i}(X^*(t))$ and use the fact that $c_i^*(t)$ realizes the maximum in (30) with $d_j = \theta/\eta_j$ and $p = \nabla v(X^*(t))$ to derive

$$\frac{1}{1-\sigma}\left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma}\right) \geq (c_i^*(t) - c_i(t))\, \frac{\partial v}{\partial x_i}(X^*(t)) \tag{33}$$

Next, observe that adding and subtracting $\big\langle (A + G^\top - \theta \quad \xi^i\eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle$ and using (32), the right-hand side in (33) equals

$$\begin{aligned} &\big\langle (A + G^\top - \theta\, \xi^i\eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \nabla v(X^*(t))\big\rangle \\ &= \big\langle X^*(t) - X(t), (A + G - \theta\eta(\xi^i)^\top)\nabla v(X^*(t))\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \nabla v(X^*(t))\big\rangle. \end{aligned} \tag{34}$$

Recalling (31) and (19), we have

$$\nabla v(X^*(t)) = b_i\langle X^*(t), \eta\rangle^{-\sigma}\eta = b_i e^{-\sigma\hat{\lambda}t}\langle x_0, \eta\rangle^{-\sigma}\eta.$$

Using this expression and the fact that $(A + G - \theta\eta(\xi^i)^\top)\eta = (\lambda - \theta(f-1))\eta$, the expression in (34) can be written as

$$= b_i\langle x_0, \eta\rangle^{-\sigma} e^{-\sigma\hat{\lambda}t}\Big[\big\langle X^*(t) - X(t), [\lambda - \theta(f-1)]\eta\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \eta\big\rangle\Big].$$

Thus, utilizing these estimates and integrating (33) on $[0, T]$ for $T > 0$, we obtain

$$\begin{aligned} \int_0^T \frac{e^{-\rho t}}{1-\sigma}\left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma}\right) dt \geq b_i\langle x_0, \eta\rangle^{-\sigma}\Big[\int_0^T e^{-(\sigma g+\rho)t}\big\langle X^*(t) - X(t), (\lambda - \theta(f-1))\eta\big\rangle dt \\ - \int_0^T e^{-(\sigma g+\rho)t}\big\langle (\dot{X}^*(t) - \dot{X}(t)), \eta\big\rangle dt\Big] \end{aligned} \tag{35}$$

and, integrating the last term by parts, the right-hand side equals

$$= b_i \langle x_0, \eta\rangle^{-\sigma} \Bigg[\int_0^T e^{-(\sigma\hat{\lambda}+\rho)t} \big\langle X^*(t) - X(t), \big(\lambda - \theta(f-1)\big)\eta\big\rangle dt +$$

$$- e^{-(\rho+\sigma\hat{\lambda})T} \langle (X^*(T) - X(T)), \eta\rangle - \int_0^T e^{-(\sigma\hat{\lambda}+\rho)t} \big\langle (X^*(t) - X(t)), (\sigma\hat{\lambda} + \rho)\eta\big\rangle dt \Bigg]$$

$$= b_i \langle x_0, \eta\rangle^{-\sigma} e^{-(\rho+\sigma g)T} \langle (X(T) - X^*(T)), \eta\rangle \geq -b_i \langle x_0, \eta\rangle^{-\sigma} e^{-(\rho+\sigma\hat{\lambda})T} \langle X^*(T), \eta\rangle , \tag{36}$$

where the last equality is a consequence of $\sigma\hat{\lambda}+\rho = \lambda-\theta(f-1)$, and the last inequality, a consequence of $\langle X(T), \eta\rangle \geq 0$, as $X(T)$ is admissible and, hence, nonnegative. Now, $e^{-(\rho+\sigma\hat{\lambda})T}\langle X^*(T), \eta\rangle = e^{-(\rho+\sigma\hat{\lambda})T} e^{\hat{\lambda}T}\langle x_0, \eta\rangle$ decreases to 0, as T tends toward $+\infty$:

$$g(1-\sigma) - \rho = -\theta < 0.$$

Thus, taking the limit as T tends toward $+\infty$ in (35)(36) (limits exist as the first integral is monotonic in T), we obtain

$$\int_0^{+\infty} e^{-\rho t} \frac{c_i^*(t)^{1-\sigma}}{1-\sigma} dt \geq \int_0^{+\infty} e^{-\rho t} \frac{c_i(t)^{1-\sigma}}{1-\sigma} dt;$$

that is, $c_i^*(t)$ is optimal.

□

A.3. **Proofs for Subsection 3.2.**

LEMMA A.1 Let $E = \xi\eta^\top$. The matrix E (respectively, $E^\top$) has an eigenvalue f with multiplicity 1, associated with the eigenvector ξ (respectively, η), and eigenvalue 0 with multiplicity $n-1$. All eigenvectors of E (respectively, $E^\top$) associated with the zero eigenvalue are orthogonal to η (respectively, ξ).

Proof of Lemma A.1. We prove the claims for E, as the arguments for $E^\top$ are similar. For any $v \in \mathbb{R}^n$, $Ev = \langle \eta, v\rangle \xi = f\xi$, implying the dimensions of the range and the kernel of E are, respectively, 1 and $n-1$. Given that $E\xi = \langle \eta, \xi\rangle \xi \neq 0$, ξ is the unique (except for multiplications by a number) eigenvector of E and the unique vector that is not in the kernel. For each vector v in the kernel of E, one has $0 = Ev = \langle \eta, v \rangle \xi$, meaning v is orthogonal to η. $\square$

Proof of Lemma 1. The proof of (i) is trivial. Accordingly, to prove (ii) we first show that any generalized eigenvector v for an eigenvalue $\lambda_i \neq \lambda$ is orthogonal to η (this property is well known but we detail it here for the reader's convenience). Thus, consider an eigenvalue $\lambda_i \neq \lambda$ (and then $i \geq 2$) and v_i as an element of the generalized eigenspace V_i. Hence, there exist a strictly positive integer m such that $(A + G^\top - \lambda_i I)^m v_i = 0$. Therefore,

$$0 = \eta^\top \left[(A + G^\top - \lambda_i)^m v_i\right] = \left[\eta^\top (A + G^\top - \lambda_i)^m\right] v_i = (\lambda - \lambda_i)^m \eta^\top \, v_i,$$

and given that $\lambda \neq \lambda_i$, then $\eta^\top \, v_i = 0$, implying η is orthogonal to v_i.

By Lemma A.1, this implies that $Ev_i = 0$, and $(A+G^\top - \theta E - \lambda_i)^m v_i = (A+G^\top - \lambda_i)^m v_i = 0$. Thus, v_i is also a generalized eigenvector for $(A + G^\top - \theta E)$ associated with the eigenvalue λ_i.

Given that we already know that $\hat{\zeta}$ is an eigenvector for $(A + G^\top - \theta E)$ associated with $\hat{\lambda}$, we only need to observe that $\{\hat{\zeta}, v_2, \ldots, v_n\}$ is a set of independent vectors. It is straightforward given that $\{v_2, \ldots, v_n\}$ are independent (they are a subset of a basis of the space) and, by the hypothesis $0 < \theta < \theta_1$, $\hat{\lambda} \neq \lambda_i$ for all $i > 1$, and $\hat{\zeta}$ is contained in a generalized eigenspace (of $(A + G^\top - \theta E)$) different from all the V_i for all $i \geq 2$ and cannot be generated by the v_i for $i \geq 2$. $\square$

The proof of Lemma A.2 and Proposition 1 are well-known facts (see, e.g., chapter 1 in Colonius and Kliemann (2014)); we provide them here for the reader's convenience.

Lemma A.2 Under the assumptions of Proposition 1, there exist continuous coefficients m_i, linear in x_0, with $\lim_{t\to\infty} m_i(t) e^{-\varepsilon t} = 0$ for all $\varepsilon > 0$ such that

$$X^*(t) = m_1(t) e^{\hat{\lambda} t} \hat{\zeta} + \sum_{i=2}^{n} e^{\mathrm{Re}(\lambda_i) t} m_i(t) v_i. \tag{37}$$

Proof of Lemma A.2. If J is the real Jordan form of the matrix $A + G^\top - \theta E$, then there exists a real invertible matrix P such that $P^{-1}(A + G^\top - \theta E)P = J$. Consequently, there exist real coefficients β_i such that

$$X^*(t) = e^{t(A+G^\top-\theta E)}x = Pe^{tJ}\left(\sum_{i=1}^{n}\langle x_0, v_i\rangle P^{-1}v_i\right) = P\sum_{i=1}^{n}\beta_i e^{Jt}P^{-1}v_i.$$

It follows then from the general theory (see, for instance, Section 1.3 of Colonius and Kliemann (2014)) that $e^{Jt}P^{-1}v_i = e^{\mathrm{Re}(\lambda_i)t}M_i(t)P^{-1}v_i$, where $M_i(t)$ is a block matrix (that is non-zero only on the Jordan block related to λ_i) whose coefficients are products of sinus and cosinus functions of t and polynomials of t with the maximum degree of the dimensions of the generalized eigenspace. As $Pe^{Jt}P^{-1}v_i$ is again an element of the generalized eigenspace associated with λ_i, it can be written as a linear combination of the eigenvectors related to the same generalized eigenspace, with the coefficient having the same described behavior as t, and then the claim follows. □

Proof of Proposition 1. The proof is entirely based on Lemma A.2 and follows from (37) once we observe that $M_i(t)$, appearing in the proof of Lemma A.2 for a simple eigenvalue, is just a real coefficient. □

A.4. **Proofs for Subsection 3.4.**

Proof of Proposition 2. If we specify condition (ii) of (24) for $x = e_j$ and $j \neq i$, we get (25); hence, it is necessary. However, if we suppose (25) is verified, given $x = \sum_{j\neq i} x_j e_j$ for some $x_j \geq 0$, we have

$$\psi_i^*(x) = \sum_{j\neq i} x_j \frac{\theta}{\eta_i}\eta_j \leq \sum_{j\neq i} x_j \frac{\frac{g_{ij}\eta_j}{\eta_i}}{\eta_i}\eta_j = \langle x, Ge_i\rangle = \left\langle G^\top x, e_i\right\rangle + \left\langle (A+G^\top)x, e_i\right\rangle,$$

wherein for the inequality, we utilized (25), and, in the last equality, we utilized $x_i = 0$; and A is diagonal, so that $\langle Ax, e_i\rangle = 0$. Therefore, (25) is also sufficient.

Further, to prove the last claim, observe that the condition (25) is equivalent to requiring that the matrix of system (23) (having nondiagonal terms $g_{ij} - \theta\eta_i\xi_j$), is indeed a Metzler matrix. That is equivalent to establishing that the system is positive; that is, it has solutions contained in the positive orthant $\mathbb{R}^n_+$ for all initial conditions $x \in \mathbb{R}^n_+$ (see, for example, Farina and Rinaldi, 2000, Chapter 2, Theorem 2, page 14). As soon as such a condition is violated, there exist trajectories of the system starting at some $x_0 \in \mathbb{R}^n_+$, which comes out of the positive orthant $\mathbb{R}^n_+$. □

Proof of Proposition 3. For simplicity, we prove the assertion for the case of all $\beta_i \equiv \beta > 0$ (for the general case the adjustment is minimal). When $\beta = 0$, the agents can only choose to fish null amounts at every node and, arguing as in Proposition 1, whatever the initial condition $x_0 \in \mathbb{R}^n_+$, the system converges to the vector ζ. By using the decomposition of Lemma A.2 and the fact that $\theta < \min\{\theta_1, \theta_2\}$, the projection of the detrended trajectory $X^*(t)e^{-\lambda t}$ on the $(n-1)$-dimensional space $\zeta^\perp$ can be decomposed into a sum of terms related to eigenvalues with a negative part. Thus (see Bitsoris, 1991), the set $E(P, d)$ is (positively) invariant for the projected system and the vector field of the velocities on the boundary of $E(P, d)$ is strictly inward.

By continuity, there exists $\bar{\beta} > 0$ such that, for any $\beta \in [0, \bar{\beta}]$, the projection of any vector field of the velocities satisfying

$$[A + G^\top - \beta]X(t) \leq \dot{X}(t) \leq [A + G^\top]X(t)$$

on the boundary of S^* is inward, and then, for any choice of the strategies in $\mathbb{A}^\beta$, the system remains in S^*. This result proves that $(S^*, \mathbb{A}^\beta)$ is a consistent couple; thus, if the equilibrium described in Theorem 1 is admissible, then it is also subgame perfect.

Given that $S^* \subseteq (0, +\infty)^n$ (and it does not touch the boundary of $\mathbb{R}^n_+$, except at the origin) there exist constants r_i such that, for any $x \in S^*$,

$$r_i \left(\sum_{j \neq i} x_j \eta_j \right) \leq x_i,$$

and, for θ and β_i such that $\theta < \beta \min_i(\eta_i / r_i)$, we have

$$\psi(x) = \frac{\theta}{\eta_i} \langle x, \eta \rangle \leq \frac{\theta}{\eta_i} \frac{x_i}{r_i} \leq \beta x_i,$$

and the set of strategies and the equilibrium described in Theorem 1 is admissible, hence yielding an MPE.

□

A.5. **Proofs for Subsection 3.5.**

Proof of Theorem 2. Assume $\hat{\psi}_j(x) = \langle w^j, x\rangle$, with $w^j \in \mathbb{R}^n_+$, $j \in N$ as a linear MPE in $\mathbb{A}$. We want to show that, necessarily, $\hat{\psi} = \psi^*$. Note that when starting at $x = e_j$, the extraction rate is $\langle w^j, e_i\rangle = w^j_i$, which implies $w^j_i \geq 0$ for all $i \in N$. We then define the square nonnegative matrices

$$W := \sum_{j\in N} e_j (w^j)^\top, \qquad W_{-i} := \sum_{j\in N, j\neq i} e_j (w^j)^\top$$

so that the stock evolves at the equilibrium with law $\dot{X} = (A + G^\top - W)X$. Given that $\hat{\psi}$ is admissible (it lies in $\mathbb{A}$ by hypothesis) at every initial stock $x_0 \in \mathbb{R}^n_+$, then $X^{\hat{\psi},x_0}(t) \geq 0$ for all $t \geq 0$, implying that $A + G^\top - W$ is a Metzler matrix (see, e.g., Farina and Rinaldi, 2000, Chapter 2, Theorem 2, page 14). Since the w^js are positive, $A + G^\top - W_{-i} = A + G^\top - W + w^i e_i$ is *a fortiori* a Metzler matrix.

We now take the standpoint of player i that assumes the other players stick to the choice $\hat{\psi}_{-i}$ and maximizes (3) for $c_i \in \mathbb{A}_i$ when subject to

$$\dot{X}(t) = (A + G^\top - W_{-i})X(t) - c_i(t)e_i, \quad X(0) = x_0$$

and under the constraint $X_i(t) \geq 0$ for all $t \geq 0$.[17]

The case $A + G - W^\top$ being irreducible. As a first step, we assume $A + G - W^\top$ is irreducible. Then, *a fortiori*, $A + G - W^\top_{-i}$ is irreducible. The Perron–Frobenius theorem implies that $A + G - W^\top$ (respectively $A + G - W^\top_{-i}$) has a simple, real eigenvalue $\hat{\lambda}$ (respectively $\hat{\lambda}^i$) strictly greater than all other eigenvalues' real parts and associated with the unique strictly positive eigenvector $\hat{\eta}$ (respectively $\hat{\eta}^i$).

[17]Note that (see, e.g., Farina and Rinaldi, 2000, Chapter 2, Theorem 2, page 14), given that $A + G - W^\top_{-i}$ is a Metzler matrix, the constraint $X_i \geq 0$ (together with the nonnegativity of the initial datum) is enough to ensure that all the components of X remain nonnegative.

The problem of agent i is associated with an HJB equation of type (14). Now, set $\hat{\eta}^i = (\hat{\eta}^i_1, \hat{\eta}^i_2, \ldots, \hat{\eta}^i_n)$, and

$$b = \left(\frac{\sigma}{\rho - \hat{\lambda}^i(1-\sigma)}\right)^{\sigma} (\hat{\eta}^i_i)^{\sigma-1}, \text{ and } \theta^i = \frac{\rho - \hat{\lambda}^i(1-\sigma)}{\sigma}. \tag{38}$$

As in the proof of Theorem 1, we can verify that a solution of this HJB equation is given by $v(x) = b(1-\sigma)^{-1} \langle x, \hat{\eta}^i \rangle^{1-\sigma}$, and (15) implies that the only candidate optimal extraction policy is

$$c_i^* = \frac{\theta^i}{\hat{\eta}^i_i} \langle \hat{\eta}^i, x \rangle .$$

Optimality can be proven using a standard verification argument, as for Theorem 1. Given that c_i^* is the only optimizer, $\hat{\psi}_i(x)$ coincides with c_i^*, implying

$$A + G - W^\top = (A + G - W^\top_{-i}) - \theta^i E_i,$$

where $E_i = \frac{1}{\hat{\eta}^i_i} \hat{\eta}^i e_i^\top$. Note: $E_i \, \hat{\eta}^i = \hat{\eta}^i$, so that $\hat{\eta}^i$ is a strictly positive eigenvector of the right- and left-hand sides of the above identity.

Given that, by Perron-Frobenius's theorem, the positive eigenvector of $(A+G-W^\top)$ is unique, necessarily,

$$\hat{\eta} \equiv \hat{\eta}^i, \quad \text{and} \quad \hat{\lambda} = \hat{\lambda}^i - \theta^i = \frac{\hat{\lambda}^i - \rho}{\sigma},$$

which can equally be proven for all $i \in F$. Hence, $\theta^i \equiv \rho - \hat{\lambda}(1-\sigma)$ for all i, and

$$W^\top = \sum_{i \in F} \theta^i E_i \Rightarrow W^\top \hat{\eta} = (\rho - (1-\sigma)\hat{\lambda}) f \; \hat{\eta},$$

so that, by difference, $\hat{\eta}$ is also a positive eigenvector of $A + G$; that is, $(A+G)\hat{\eta} = (\rho - (1-\sigma)\hat{\lambda}) f \hat{\eta} + \hat{\lambda}\hat{\eta}$. However, necessarily, $\hat{\eta} \equiv \eta$, and $(\rho - (1-\sigma)\hat{\lambda}) f + \hat{\lambda} \equiv \lambda$. Therefore, $\lambda = \theta^i f + \frac{\rho - \theta^i}{1-\sigma}$, and $\theta^i = \frac{\rho + (\sigma-1)\lambda}{1+(\sigma-1)f} = \theta$. Thus, we have proven that $\psi^* \equiv \hat{\psi}$.

The case $A+G-W^\top$ being reducible. We now consider the case of $A+G-W^\top$ being reducible. For brevity, we set $M = A + G^\top - W$, and $M_{-i} = A + G^\top - W + e_i {w^i}^\top$. Barring a permutation (i.e., a change in nodes name), we can assume that $M^\top = A + G - W^\top$ is in Frobenius form (see (1.7.1) page 38 of Bapat and Raghavan, 1997), reading as

$$M^\top = \begin{bmatrix} M_1^\top & M_{21}^\top & \dots & M_{K1}^\top \\ 0 & M_2^\top & \dots & M_{K2}^\top \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & M_K^\top \end{bmatrix},$$

with irreducible submatrices $M_k^\top$ on the diagonal (some $M_{Ki}^\top$ can contain zeros). Given that G is strictly positive, and there is no extraction in reserves, all the matrix elements from a reserve to any other location must be strictly positive. Thus, reserves are among the locations associated with $M_K^\top$, although the same block may partly also refer to fishing locations.

As before, we denote by $\hat{\lambda}$ the dominant eigenvalue[18] of $M^\top$, by $\hat{\eta}$ one of the associated eigenvectors, by $\hat{\lambda}^i$ the dominant eigenvalue of $M_{-i}^\top$, and by $\hat{\eta}^i$ one of the associated eigenvectors. Moreover, we denote by λ_k the dominant eigenvalue of M_k for any $k = 1, \dots, K$.

The rest of the proof is divided into two steps.

Step 1: if there exists $i \in F$ for which $\hat{\eta}^i > 0$, then the only linear equilibrium is the one described in Theorem 1. Indeed, arguing as in the case of an irreducible M, we again obtain $w^i = (\theta^i/\hat{\eta}^i_i)\hat{\eta}^i$, where θ_i is given by (38), and $\hat{\eta}^i$ is an eigenvector of the matrix $M^\top$. Given that $\hat{\eta}^i > 0$ by assumption, it coincides with the unique dominant eigenvector of $M^\top$ (Theorem 11, page 36 of Farina and Rinaldi (2000)) and in particular with $\hat{\eta}$.

We look now at the behavior of other agents. We call agent-jk any agent j fishing (at node j) in the subset of nodes k. Its extraction vector at the given equilibrium is denoted by $w^j = (w_{jk}^{(1)}, w_{jk}^{(2)}, \dots, w_{jk}^{(K)})^\top$, where $w_{jk}^{(h)}$ is a vector with as many coordinates as the dimension of block M_h. We first look at a possible agent in one

[18] Even if not irreducible, $M^\top$ is a Metzler matrix, and we can apply a weak form of the Perron-Frobenius theorem as Theorem 1.7.3 of Bapat and Raghavan (1997), jointly with Footnote 6, to imply that the spectral radius of the matrix is an eigenvalue, dominant (its real part is higher than the real part of any other eigenvalue), and associated with a nonnegative eigenvector, possibly not unique.

of the nodes related to K. First, observe that (given that KK is the down-right block, and we already proved that the unique dominant eigenvector of $M^\top$ is strictly positive), one must have $\lambda_K = \hat{\lambda}$. Moreover, $\hat{\lambda}$ is the maximum of all λ_k; thus, $\lambda_K \geq \lambda_k$ for all k. Now the control problem for agent-jK is associated with the matrix

$$M_{-j}^\top = \begin{bmatrix} M_1^\top & M_{21}^\top & \dots & M_{K1}^\top \\ 0 & M_2^\top & \dots & M_{K2}^\top \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & M_K^\top \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 & w_{jK}^{(1)} \\ 0 & \dots & 0 & w_{jK}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & w_{jK}^{(K)} \end{bmatrix}.$$

The Perron's eigenvalue of $M_K^\top + w_{jK}^{(K)}$ is higher than λ_K and all λ_k terms. The dominant eigenvalue of $M_{-j}^\top$ is then strictly positive, unique, and associated with the eigenvalue of $M_K^\top + w_{jK}^{(K)}$. Arguing again as above, one computes the optimal closed-loop control of agent-jK and verifies that $w^j = (\theta^j/\hat{\eta}_j^j)\hat{\eta}$, (where $\hat{\eta}$ is the Frobenius eigenvector of $M^\top$).

We show now that the same holds for any other agent-jk, $k \neq K$. We first show that there exists a unique eigenvector associated with $\hat{\lambda}^j$, and it is strictly positive. We consider the matrix

$$\begin{bmatrix} a_{jk} & m_{Kjk}^\top \\ g_{jkK} & M_K^\top \end{bmatrix},$$

which is the proper principal submatrix of $M_{-j}^\top$ obtained by removing all rows and corresponding columns that are not in the K block and are not jk. Its dominant eigenvalue is strictly greater than λ_K and, at the same time (see Theorem 1.7.4 of Bapat and Raghavan, 1997), smaller than $\hat{\lambda}^j$; thus, $\hat{\lambda}_j > \hat{\lambda} = \lambda_K$. This result implies that all matrices of type $[\hat{\lambda}^j I - M_k^\top]$ have a strictly positive inverse (see Theorem 1.7.2 page 35 of Bapat and Raghavan, 1997). This fact is sufficient to show that all the components of any eigenvector $\hat{\eta}^j$ associated with $\hat{\lambda}^j$ (which is ex-post unique; see Theorem 11, page 36 of Farina and Rinaldi, 2000) are strictly positive.

For the components related to the block K, we have

$$\hat{\eta}^j_K = \hat{\eta}^j(j)[\hat{\lambda}_j I - M^\top_{KK}]^{-1} g_{jkK}, \tag{39}$$

where $\hat{\eta}^j_K$ is the part of the eigenvector $\hat{\eta}^j$ corresponding to the areas in K, $\hat{\eta}^j_K(j)$ is the j-th component of the same eigenvector, and g_{jkK} is the vector of inflows from the K part. Therefore, prices in this subset are either all positive or zero. Proceeding recursively, the same alternative occurs for all prices in the other blocks of the matrix. Thus, eventually $\hat{\eta}_j > 0$, which is true for all agents.

With the same argument as in the irreducible case, we compute the optimal closed-loop controls of all agents and verify first that, for all j, w^j is indeed the Frobenius's eigenvector of $M^\top$, and, except for multiplication factors, it is the unique strictly positive eigenvector of $A+G$. The uniqueness of the multiplication factors follows the proof of Theorem 1, completing the proof of the uniqueness of the linear equilibrium.

Step 2: There exists an agent i for which $\hat{\eta}^i > 0$. In the case in which agents concentrate in the subset K of nodes, the matrix is irreducible. Then, without loss of generality, we assume agents are distributed also in regions other than those in the subset K. If $\lambda_K > \lambda_k$, then, arguing as in Step 1, all $\hat{\eta}^i$s are strictly positive and coincide with $\hat{\eta}$. In the opposite case, there exists k^*, with $k^* \neq K$ such that $\lambda_{k^*} \geq \mathrm{Re}\lambda_k$ for all $k = 1, \dots, K$.

Now consider an agent-ik^*; that is, operating in region i and belonging to subset k^*. Arguing as in Step 1, we see that the matrix $M_{-i} = M^\top_{k^*} + w^{(k^*)}_{ik^*} e^\top_i$ has eigenvalue $\hat{\lambda}^i$ with $\hat{\lambda}^i > \lambda_{k^*}$. Thus, all matrices of type $[\hat{\lambda}^i I - M^\top_k]$ with $k \neq k^*$ have a positive inverse. In particular, (39) implies

$$\hat{\eta}^i_K = \hat{\eta} i(i)[\hat{\lambda}_i I - M^\top_K]^{-1} g_{ik^*K}.$$

Substituting backward into the eigenvector equation for the $K-k^*-1$ block, we find a similar equation. Proceeding iteratively until we reach k^*, we see that all prices in the areas "downstream" of k^* are either all positive if the component of the eigenvector at ik^* is positive, or all zero if the same component is zero. However, this component cannot be zero given that, otherwise, one would have to find a non-strictly positive eigenvector of the irreducible matrix

$$M_{k^*k^*}^\top + w_{ik^*}^{(k^*)} e_{ik^*}^\top.$$

Given that the components of the eigenvector at ik^* are positive, all prices "upstream" of k^* are also positive. Therefore, we have an agent with positive prices. □

Proof of Theorem 3 (Sketch). The existence part of the statement is proved in Proposition 3. We sketch here the proof of uniqueness, using the arguments of the proof of Theorem 2. We assume a linear MPE; that is, a set of strategies of the form $\hat{\psi}_j(x) = \langle w^j, x\rangle$, with $w^j \in \mathbb{R}^n_+$, $j \in N$ and such that $\hat{\psi}_j(x) \le \beta_j x_j$ for $j \in N$. As for the proof of Proposition 3 we prove the assertion for the case of all $\beta_j = \beta$ (for the general case the adjustment is minimal).

First, arguing as in the proof of Proposition 3, we derive that the trajectories are contained in a cone in $(0, +\infty)^n$. Thus, there exist constants $s^j_m > 0$ (which depend only on the cone structure that can be chosen independently of β) such that

$$x_j \le s^j_m x_m.$$

The constraint $\hat{\psi}_j(x) = \langle w^j, x\rangle \le \beta x_j$ implies $w^j_m x_m \le \beta x_j$ for all $m \in N$ (recall that w^j_m are nonnegative, as shown along the proof of Theorem 2). Thus,

$$w^j_m \le \beta \frac{x_j}{x_m} \le \beta s^j_m.$$

If one chooses $\beta > 0$ that is small enough, by continuity, since the matrix $A+G$ has a simple dominant eigenvalue associated with a strictly positive eigenvector, the same properties hold for the matrix $A + G - W^\top$ and all the $A + G - W_i^\top$ matrices. Then, after choosing a θ small enough to ensure (as in the proof of Proposition 3) that the candidate equilibrium is admissible, we can argue exactly as in the proof of Theorem 2 in the case where $A + G - W^\top$ is irreducible. □

A.6. **Proofs for Section 4.**

Proof of Proposition 5. To prove (i), we extend to Metzler matrices the well-known result that the largest eigenvalue of an irreducible positive matrix is an increasing function of its elements (see e.g., Berman and Plemmons, 1994, Chapter 2, Corollary 1.5). We consider $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ two sets of reproduction rates, the matrices $A+G$ and $A'+G$ of the associated systems, and their maximal eigenvalues λ and λ'. If $\Gamma_i \leq \Gamma'_i$ for every i, then clearly $A+G \leq A'+G$. Hence, our thesis is $\lambda \leq \lambda'$.

Given that there exists a constant c such that $A+G+cI$ and $A'+G+cI$ are positive, I being the identity matrix, and $A+G+cI \leq A'+G+cI$, then the associated maximal eigenvalues bear the same order $\lambda(A+G+cI) \leq \lambda(A'+G+cI+cI)$. Given that $\lambda(A+G+cI) = \lambda + c$, $\lambda(A+G+cI) = \lambda' + c$; moreover, $\hat{\lambda}$ is an increasing function of λ (see (20)), the proof of (i) is complete. Statements (ii) and (iii) hold by direct calculations, in all subcases (a) (b) (c)

$$\frac{d\hat{\lambda}}{d\lambda} = \frac{1}{(\sigma-1)f+1} > 0, \quad \frac{dg}{df} = -\frac{\rho + \lambda(\sigma-1)}{(1+\lambda(\sigma-1))^2} < 0.$$

□

Proof of Proposition 6. We first check the effect of an ϵ increase in g_{ij}, with $i \neq j$, on the value of λ. Accordingly, fix $\epsilon > 0$ and define $M_{ij} := (e_i\, e_j^\top + e_j\, e_i^\top) - (e_i\, e_i^\top + e_j\, e_j^\top)$. Note that the system matrix changes from $A+G$ to $A+G+\epsilon M_{ij}$. This last matrix can be written as the sum of two Metzler matrices

$$A+G+\epsilon M_{ij} = [A - \epsilon(e_i\, e_i^\top + e_j\, e_j^\top)] + [G + \epsilon(e_i\, e_j^\top + e_j\, e_i^\top)],$$

so that it is a Metzler matrix. Moreover, M_{ij} is a negative-semidefinite matrix, so that $\langle x, M_{ij}x\rangle \leq 0$ for all $x \in \mathbb{R}^n$. Since $A+G$ is irreducible, if ϵ is small enough the matrix $A+G+\epsilon M_{ij}$ is irreducible. We denote by η_ϵ its Perron-Frobenius eigenvector of norm 1 (see Footnote 18), and by λ_ϵ the associated Perron-Frobenius eigenvalue. Given that the network matrix is symmetric, we can utilize the variational characterization of eigenvalues (see for instance Corollary III.1.2 of Bhatia, 2013), so that

$$\max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G+\epsilon M_{ij})x\rangle}{|x|^2} = \lambda_\epsilon = \frac{\langle \eta_\epsilon, (A+G+\epsilon M_{ij})\eta_\epsilon\rangle}{|\eta_\epsilon|^2}$$
$$\leq \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G)x\rangle}{|x|^2} \; + \; \epsilon\frac{\langle \eta_\epsilon, M_{ij}\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \leq \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G)x\rangle}{|x|^2} = \lambda.$$

This result implies $\frac{d\lambda}{dg_{i,j}} \leq 0$ and (i) is proved. Thus, (ii) follows from (i) and (20). □

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