DECISION-MAKING MODELS IN MARKETING: DYNAMIC OPTIMIZATION AND DIFFERENTIAL GAMES [1]

Igor Bykadorov [2], *Andrea Ellero* [3], *Elena Moretti* [4]

Abstract

The purpose of this paper is to show some economic outcomes of the application of optimal control to marketing. We present two dynamic models. In the first one we consider a firm that seeks to reach a fixed level of goodwill at the end of the selling period, with the minimum total expenditure in promotional activities. The firm can only control the communication expenditure rate. In the second one we consider a vertical control distribution channel where a manufacturer sells a single kind of good to a retailer. The manufacturer's profit maximization is considered in an optimal control model where the manufacturer's control is the discount on wholesale price (trade discount). The optimal discount policy of the manufacturer turns out to depend on the efficiency of the retailer and his sale motivation. Finally we also consider the manufacturer-retailer relationship in a differential game framework: the manufacturer's control is again the trade discount while the retailer controls the pass-through to the market.

Keywords: optimal control, advertising, sales promotions, communications, sales motivation, vertical channel, trade discount, differential game.

---

[1] *This research has been supported by Università Ca' Foscari di Venezia, Research program of national interest MIUR PRIN-2005 "Retailer and supplier performance in the distribution channel: optimal control and efficiency evaluation", the Counsil for Grants (under RF President) and state Aid of Fundamental Science Schools (grant no. NSh-4999.2006.6).*

[2] Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, 4 Acad. Koptyug avenue, 630090, Novosibirsk, Russia, e-mail: bykad@math.nsc.ru

[3] Dipartimento di Matematica Applicata, Università Ca' Foscari di Venezia, Dorsoduro 3825/E, 30123, Venezia, Italia, e-mail: ellero@unive.it

[4] Dipartimento di Matematica Applicata, Università Ca' Foscari di Venezia, Dorsoduro 3825/E, 30123, Venezia, Italia, e-mail: emoretti@unive.it

1 Introduction

In this paper we present two types of dynamic models for marketing.

In section 2 we consider a firm that seeks to reach a fixed level of goodwill at the end of the selling period, with minimum total expenditure in promotional activities. We consider the linear optimal control problem faced by the firm which can only control the communication expenditure rate; communication is performed by means of advertising and promotion. Goodwill and sales levels are considered as state variables and word-of-mouth effect and saturation aversion are taken into account. We show that the structure of the optimal communication depends on the relation between the negative factors for the firm (saturation aversion and goodwill decay) and the the positive ones (word-of-mouth and goodwill productivity). Moreover, we can understand that the structure of the optimal communication is bang-bang and identify the number of switches, even if the total description of the optimal communication can be done only numerically.

In section 3 we consider a vertical control distribution channel where a manufacturer sells a single kind of good to a retailer. We assume that a discount in wholesale price increases the retailer's sales motivation and consequently it increases sales. We study both the manufacturer's and retailer's profit maximization problems as optimal control models. The manufacturer's control is the discount on wholesale price (trade discount) allowed to the retailer. The retailer's control is the part of trade discount transferred by the retailer to the market consumers (pass-through). The optimal control of manufacturer's and retailer's profit via trade discount and pass-through is embedded in a differential game framework.

Let us remark that our aim here is to give an idea of the models and of the results obtained. So we omit the proofs of the results, which are rather technical, or give only "the sketch" of the proof. The detailed proofs can be found in [2], [3], [4], [5], [6], [7].

2 Minimization of communication expenditure

The problem of planning media schedules and communication mix expenditure over time has received growing attention in the recent past and a number of aggregate advertising response models have been proposed in literature since the pioneering works of Vidale and Wolfe [22] and Nerlove and Arrow [18]. The key idea of Nerlove and Arrow was to take into account explicitly the goodwill level reached by a firm or product; the goodwill depends on the expenditure in advertising and is subject to a decay due to the forgetting effect. More precisely the dynamics in the model of Nerlove and Arrow is described by the linear differential equation

$$\dot{A}(t) = a(t) - \delta A(t) \ ,$$

where $A(t)$ is the goodwill level at time t, $a(t)$ is the advertising spending rate at time t and δ is the goodwill decay rate. A number of generalizations of the model, with suitably defined profit functions to be maximized, can be found in literature (see e.g. Feichtinger, Hartl and Sethi [10]).

Here we consider a firm that sells goods and seeks for the optimal communication plan over a finite time horizon: this leads to a linear optimal control model.

Similar models were recently proposed by Favaretto and Viscolani [9], Buratto and Favaretto [1], Funari and Viscolani [11]. In particular in [9] an optimal control model for production, advertising and selling of seasonal goods is considered where production and advertising take place in a first time period while in a second consecutive time period the firm can sell the good, continuing the advertising activity.

In our model we focus on the interaction of different promotional activities during the selling period. In fact we take into account not only advertising expenditure but also other important features of the communication mix, like sales promotion and word-of-mouth. Moreover we include in the model a market saturation effect.

We assume that the aim of the firm is to reach at the end of the selling period a level of goodwill previously defined by the management: this can be useful, for example, when reaching high levels of goodwill allows to exploit hysteresis properties of the response function [15], [14], so that it is possible to keep the level of goodwill with low economical effort. At the same time we assume that the firm requires to sell its whole inventory.

To reach its targets the firm spends in promotional activities but since, as it is well known, many brands are overspending in advertising, a careful minimization of costs is required and the firm tries to minimize the total expenditure in communication.

To be more precise, we assume that the communication expenditure rate is the only control allowed to the firm and that communication is performed by means of advertising and sales promotion. In the linear optimal control model, goodwill dynamics depends not only on the advertising effort but also on the sales level due to the effect of word-of-mouth.

High levels of goodwill improve sales, of course, but selling activity depends directly also on sales promotion and on sales level itself. In particular, the increasing level of goods already sold will reduce sales speed due to the progressive saturation of the market. Sales level changes instead, depend directly on sales promotions efforts and on the reached sales level and take into account congestion aversion.

2.1 Formulation of the Problem

As mentioned in the previous section, we propose a linear optimal control problem to model the dynamics of selling and communication activities of a firm. Of course, linearity is a strong assumption but, since we consider a firm that sells seasonal goods, the time period in which the dynamics evolves is limited and short enough so that a linear model can be considered a sufficiently good approximation of reality.

Since the selling period is short we also assume that the marginal effects of communication activities are constant and positive both with respect to the sales and to the goodwill of the firm. The total expenditure rate in communication is bounded and divided *a priori* by the management into two parts, one for advertising and one for promotion.

The motion equation we consider for the goodwill generalizes the one proposed by Nerlove and Arrow [18] and is given by

$$\dot{A}(t) = \varphi x(t) - \delta_A A(t) + \gamma_A \rho a(t)$$

where

$$\begin{aligned}
A(t) &= \text{goodwill level at time } t, \\
x(t) &= \text{sales level at time } t, \\
a(t) &= \text{communication expenditure rate at time } t, \\
\delta_A &= \text{goodwill decay parameter, } \delta_A > 0, \\
\gamma_A &= \text{advertising productivity in terms of goodwill, } \gamma_A > 0, \\
\rho &= \text{weight of the total expenditure rate devoted to advertising, } \rho \in [0,1]
\end{aligned}$$

and φ is the word-of-mouth productivity in terms of goodwill. Thus the word-of-mouth effect increases the goodwill rate whenever $\varphi > 0$ while a negative word-of-mouth effect corresponds to $\varphi < 0$ [5]. In the following we will restrict our attention to the case of a favorable word-of-mouth, i.e. $\varphi > 0$ [6].

The sales level dynamics will be defined by the equation

$$\dot{x}(t) = -\theta x(t) + \delta_x A(t) + \gamma_x (1-\rho) a(t)$$

where

$$\begin{aligned} \delta_x &= \text{goodwill productivity in terms of sales, } \delta_x > 0, \\ \gamma_x &= \text{promotion productivity in terms of sales, } \gamma_x > 0, \end{aligned}$$

and $\theta > 0$ is a saturation aversion parameter: in fact this way the sales rate decreases as the cumulative sales increase, modeling the market saturation effect. Factor $(1-\rho)$ is the part of the total expenditure rate devoted to promotion.

During the selling period the firm requires to reach a fixed goodwill level $\tilde{A}$ starting from the initial level $\overline{A}$ and to sell the total inventory $\overline{m}$.

This way the following optimal control problem can be formulated [3]

$$\begin{aligned} P \ : \ \text{minimize} \quad & \int_{t_1}^{t_2} a(t)\,\mathrm{d}t, \\ \text{subject to} \quad & \dot{x}(t) = -\theta x(t) + \delta_x A(t) + \gamma_x (1-\rho) a(t) \\ & \dot{A}(t) = \varphi x(t) - \delta_A A(t) + \gamma_A \rho a(t) \\ & x(t_1) = 0, \qquad x(t_2) = \overline{m}, \\ & A(t_1) = \overline{A}, \qquad A(t_2) = \tilde{A}, \\ & a(t) \in [0, \overline{a}], \end{aligned}$$

where $[t_1, t_2]$ is the selling period and $\overline{a} > 0$ is the upper bound for the communication expenditure rate.

2.2 Preliminary assumptions

We will assume that problem P satisfies the general position condition (GPC) (see e.g. [20], p.166), which guarantees the uniqueness of the solution, if any.

---

5) The role of parameter φ is rather similar to the seller's *reputation* in Spremann's model [21].

6) This is only done in order to restrict the number of sub-cases to consider; negative values of word-of-mouth are of course possible in practice, e.g. when selling a low quality product which is initially perceived by the market, due to unfair advertising, as a high quality product.

The "regularity" of the problem required by this assumption is considered quite natural in optimal control since in practical problems non-zero coefficients are known only with some approximation.

Remark that our assumption requires in particular that $\theta\delta_A - \varphi\delta_x \neq 0$ and we will see that $\theta\delta_A - \varphi\delta_x = 0$ is a threshold for the qualitative properties of the optimal control of problem P.

Another technical assumption that we will adopt throughout the paper is that the control is continuous from the left and continuous at the end point t_2 (see e.g. [20], p.73).

2.3 Types of optimal control

The system of motion equations of problem P can be rewritten as

$$\dot{X}(t) = QX(t) + B(a(t)), \tag{1}$$

where

$$X(t) = \begin{pmatrix} x(t) \\ A(t) \end{pmatrix}, \quad Q = \begin{pmatrix} -\theta & \delta_x \\ \varphi & -\delta_A \end{pmatrix}, \quad B(a(t)) = a(t) \begin{pmatrix} \gamma_x(1-\rho) \\ \gamma_A\rho \end{pmatrix}.$$

To apply the Pontryagin's Maximum Principle [19] we need to compute the eigenvalues of matrix $-Q^T$ which are

$$\lambda_1 = \frac{\theta + \delta_A - \sqrt{(\theta - \delta_A)^2 + 4\varphi\delta_x}}{2}, \quad \lambda_2 = \frac{\theta + \delta_A + \sqrt{(\theta - \delta_A)^2 + 4\varphi\delta_x}}{2}.$$

Remark that, since we assumed $\varphi > 0$, we have $\lambda_1, \lambda_2 \in \mathbb{R}$, moreover $\lambda_2 > 0$, $\lambda_2 > \lambda_1$, and the sign of λ_1 coincides with the sign of $\theta\delta_A - \varphi\delta_x$ (7).

Due to the GPC assumption and since the eigenvalues are real, the optimal control is "bang-bang" and the number of switches in the optimal control cannot be more than two [20].

The following Proposition explains how the sign of λ_1 determines the type of optimal control.

Proposition 2.1 *(See [3].) If $\theta\delta_A < \varphi\delta_x$ (i.e. $\lambda_1 < 0$) then the optimal communication policy can only be of type*

$$a^*(t) = \begin{cases} \overline{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, \tau_2) \\ \overline{a}, & t \in (\tau_2, t_2] \end{cases} \tag{2}$$

7) This implies that, under GPC assumption, $\lambda_1 \neq 0$; the special case in which $\lambda_1 = 0$ is considered, under some more restrictive hypotheses, in [9].

with switch times τ_1, τ_2 such that $t_1 \le \tau_1 \le \tau_2 \le t_2$;
if $\theta\delta_A > \varphi\delta_x$ then the optimal communication policy can only be of type

$$a^*(t) \;=\; \begin{cases} 0, & t \in [t_1, \tau_1) \\ \overline{a}, & t \in (\tau_1, \tau_2) \\ 0, & t \in (\tau_2, t_2] \end{cases} \tag{3}$$

with $t_1 \le \tau_1 \le \tau_2 \le t_2$.

From Proposition 2.1 we have

Corollary 2.1 *If problem P has some admissible control then one and only one of the following communication policies a^* is optimal:*
"ALTERNATE $\overline{a} - 0 - \overline{a}$ communication": some τ_1, τ_2 exist such that

$$a^*(t) \;=\; \begin{cases} \overline{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, \tau_2) \\ \overline{a}, & t \in (\tau_2, t_2] \end{cases} \quad ;$$

"ALTERNATE $0 - \overline{a} - 0$ communication": some τ_1 and τ_2 exist such that

$$a^*(t) \;=\; \begin{cases} 0, & t \in [t_1, \tau_1) \\ \overline{a}, & t \in (\tau_1, \tau_2) \\ 0, & t \in (\tau_2, t_2] \end{cases} \quad ;$$

"EARLY communication": some τ_1 exists such that

$$a^*(t) \;=\; \begin{cases} \overline{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, t_2] \end{cases} \quad ;$$

"LATE communication": some τ_1 exists such that

$$a^*(t) \;=\; \begin{cases} 0, & t \in [t_1, \tau_1) \\ \overline{a}, & t \in (\tau_1, t_2] \end{cases} \quad ;$$

"MAXIMUM communication": $a^*(t) = \overline{a} \quad \forall t \in [t_1, t_2]$;
"NO communication": $a^*(t) = 0 \quad \forall t \in [t_1, t_2]$.

2.4 Admissible optimal controls

In order to find admissible optimal controls we have to consider system (1) with t belonging to some interval $[\, t', t'' \,] \subseteq [t_1,\ t_2]$ and the control $a(t)$ constant and either equal to zero or equal to $\overline{a}$, as stated in Proposition 2.1.

Let S be a matrix, whose i-th column is an eigenvector of matrix Q (see system (1)), corresponding to eigenvalue $-\lambda_i$. For example, matrix S can be written as

$$S = \begin{pmatrix} \delta_x & \delta_x \\ \theta - \lambda_1 & \theta - \lambda_2 \end{pmatrix}.$$

Remark that S is nonsingular since $\lambda_1 \neq \lambda_2$. Let us define

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = -S^{-1}QS \ ,$$

which is a nonsingular matrix due to the GPC assumption. Further, define

$$D(t) = e^{t\Lambda}\Lambda^{-1}S^{-1}B(\overline{a}).$$

Then the solution of (1) on the interval $[\ t', t''\ ]$ can be written as

$$X \ = \ \begin{cases} Se^{(t'-t)\Lambda}S^{-1}X(t'), & \text{if } a(t) = 0 \\ Se^{-t\Lambda}\{e^{t'\Lambda}S^{-1}X(t') - D(t') + D(t)\}, & \text{if } a(t) = \overline{a} \end{cases} \quad . \tag{4}$$

At this point, using (4) we can write the dynamics of the system depending on the optimal control. Moreover, requiring continuity of the optimal trajectories, we can specify the conditions on switch times that allow to have an optimal control, as given in Proposition 2.1. To simplify notation we define the following constant vector

$$G = e^{t_2\Lambda}S^{-1}X(t_2) - e^{t_1\Lambda}S^{-1}X(t_1) \quad .$$

Proposition 2.2 *(See [3].) The optimal trajectory for problem P is:*

a.

$$X^* \ = \ \begin{cases} Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(t)\}, & t \in [t_1, \tau_1] \\ Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(\tau_1)\}, & t \in [\tau_1, \tau_2] \\ Se^{-t\Lambda}\{e^{t_2\Lambda}S^{-1}X(t_2) - D(t_2) + D(t)\}, & t \in [\tau_2, t_2] \end{cases}$$

with τ_1, τ_2 such that $t_1 \leq \tau_1 \leq \tau_2 \leq t_1$ and

$$D(\tau_1) - D(\tau_2) = G + D(t_1) - D(t_2) \ , \tag{5}$$

if the optimal control is "alternate $\overline{a} - 0 - \overline{a}$ communication" or "early communication" ($\tau_2 = t_2$) or "maximum communication" ($\tau_1 = \tau_2 = t_2$);

b.

$$X^* = \begin{cases} Se^{(t_1-t)\Lambda}S^{-1}X(t_1), & t \in [t_1, \tau_1] \\ Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(\tau_1) + D(t)\}, & t \in [\tau_1, \tau_2] \\ Se^{(t_2-t)\Lambda}S^{-1}X(t_2), & t \in [\tau_2, t_2] \end{cases}$$

and τ_1, τ_2 *such that* $t_1 \le \tau_1 \le \tau_2 \le t_1$ *and*

$$D(\tau_2) - D(\tau_1) = G \quad , \tag{6}$$

if the optimal control is "alternate $0 - \overline{a} - 0$ *communication" or "late communication"* ($\tau_2 = t_2$) *or "no communication"* ($\tau_1 = \tau_2 = t_2$)*.*

By means of Proposition 2.2 we can outline how any instance of problem P can be solved. To this aim let us first remark that if $\tau_2 = t_2$ then equation (5) becomes

$$D(\tau_1) = G + D(t_1) \tag{7}$$

while (6) writes

$$D(\tau_1) = -G + D(t_2) \quad . \tag{8}$$

Moreover, if $\tau_1 = \tau_2 = t_2$ then (5) becomes

$$G = D(t_2) - D(t_1) \tag{9}$$

and (6) becomes

$$G = (0,\ 0)^T \quad . \tag{10}$$

Now it is possible to describe **an algorithm** to solve problem P.

Compute λ_1.
If $\lambda_1 < 0$ then check conditions (9) and (10). If one of them is satisfied we have maximum or no communication, respectively, and the algorithm stops.
Otherwise the systems of two equations (7), (8) and (5) must be considered. If one of them has solution belonging to the interval $[t_1, t_2]$ then the corresponding optimal control is determined, otherwise problem P has no solution: this way the problem is completely solved.
Remark that while systems (7) and (8) can be explicitly solved, system (5) must be solved numerically. Anyway, since at most one of them has solution, not necessarily all of them must be solved.
If $\lambda_1 > 0$ then the procedure is the same but one has to consider system (6) instead of (5).

2.5 Parametric analysis

In this section we show how optimal communication policies vary depending on the boundary conditions.

To this aim we look for a graphic representation of the sets of boundary values of total inventory $\overline{m}$, initial and final goodwill levels $\overline{A}$ and $\tilde{A}$, for which the structure of the optimal control is of the same kind.

We will use the conditions on τ_1 and τ_2 given in Proposition 2.2. More precisely: if $\lambda_1 < 0$ then in "alternate $\overline{a} - 0 - \overline{a}$ communication" case some τ_1 and τ_2 must exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (see (5) and (4))

$$(e^{\tau_1 \Lambda} - e^{\tau_2 \Lambda})\Lambda^{-1} S^{-1} B(\overline{a}) = G + (e^{t_1 \Lambda} - e^{t_2 \Lambda})\Lambda^{-1} S^{-1} B(\overline{a}); \tag{11}$$

while if $\lambda_1 > 0$ then in "alternate $0 - \overline{a} - 0$ communication" case some τ_1 and τ_2 exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (see (6) and (4))

$$(e^{\tau_2 \Lambda} - e^{\tau_1 \Lambda})\Lambda^{-1} S^{-1} B(\overline{a}) G. \tag{12}$$

The space in which we give the parametric representation of the types of optimal control of problem P is obtained transforming the space of parameters $\overline{m}$, $\overline{A}$, $\tilde{A}$, in a suitable two dimensional space.

In order to define this transformation let

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = S^{-1} B(\overline{a}) \ , \qquad\qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = G \ .$$

It is easy to show that, under GPC assumption, $d_i \neq 0, i = 1, 2$, therefore we can also define

$$k_i = \frac{g_i}{d_i} \ . \tag{13}$$

This way equation (11) can be rewritten as

$$e^{\lambda_i \tau_1} - e^{\lambda_i \tau_2} = \lambda_i k_i + e^{\lambda_i t_1} - e^{\lambda_i t_2} \ , \ i = 1, 2 \ ,$$

and (12) becomes

$$e^{\lambda_i \tau_2} - e^{\lambda_i \tau_1} = \lambda_i k_i \ , \ i = 1, 2 \ .$$

We will work now in the space of k_1, k_2 (the definition of k_i see in (13)), it means that we will not work in the three-dimensional space of variables $\overline{A}$, $\overline{m}$ and $\tilde{A}$, but "in terms" of them since k_1 and k_2 depend linearly on $\overline{A}$, $\overline{m}$, $\tilde{A}$.

Let us define

$$f_-(k_1,k_2) = \frac{1}{\lambda_1}\ln(e^{\lambda_1 t_2} - \lambda_1 k_1) - \frac{1}{\lambda_2}\ln(e^{\lambda_2 t_2} - \lambda_2 k_2) \quad ,$$

and

$$f_+(k_1,k_2) = \frac{1}{\lambda_1}\ln(e^{\lambda_1 t_1} + \lambda_1 k_1) - \frac{1}{\lambda_2}\ln(e^{\lambda_2 t_1} + \lambda_2 k_2) \quad .$$

It is now possible to state the following Theorem.

Theorem 2.1 *(See [3].) If* $f_+(k_1,k_2) > 0$ *or* $f_-(k_1,k_2) > 0$ *then there is no feasible control for problem* P*; otherwise the optimal control of* P *is:*
"alternate $\overline{a} - 0 - \overline{a}$ *communication" if* $f_+(k_1,k_2) < 0$, $f_-(k_1,k_2) < 0$ *and* $\lambda_1 < 0$*;*
"alternate $0 - \overline{a} - 0$ *communication" if* $f_+(k_1,k_2) < 0$, $f_-(k_1,k_2) < 0$ *and* $\lambda_1 > 0$*;*
"early communication" if $f_+(k_1,k_2) = 0$ *and* $f_-(k_1,k_2) < 0$*;*
"late communication" if $f_+(k_1,k_2) < 0$ *and* $f_-(k_1,k_2) = 0$*;*
"maximum communication" if $f_+(k_1,k_2) = 0$, $f_-(k_1,k_2) = 0$ *and* $h_1 = h_2 = 0$*;*
"no communication" if $f_+(k_1,k_2) = 0$, $f_-(k_1,k_2) = 0$ *and* $k_1 = k_2 = 0$*.*

From Theorem 2.1 we have that the set

$$V_1 = \{(k_1,k_2) \mid f_+(k_1,k_2) < 0,\ f_-(k_1,k_2) < 0\}$$

is the region in the space of parameters k_1 and k_2 in which the optimal control of problem P is alternate communication, "alternate $\overline{a} - 0 - \overline{a}$ communication" if $\lambda_1 < 0$, "alternate $0 - \overline{a} - 0$ communication" if $\lambda_1 > 0$.

The sets corresponding to the other kinds of communication are

$V_2 = \{(k_1,k_2) \mid f_+(k_1,k_2) = 0,\ f_-(k_1,k_2) < 0\}$ early communication curve ,

$V_3 = \{(k_1,k_2) \mid f_+(k_1,k_2) < 0,\ f_-(k_1,k_2) = 0\}$ late communication curve ,

$V_4 = \{(\frac{1}{\lambda_1}(e^{\lambda_1 t_2} - e^{\lambda_1 t_1}),\ \frac{1}{\lambda_2}(e^{\lambda_2 t_2} - e^{\lambda_2 t_1}))\}$ maximum communication point ,

$V_5 = \{(0,0)\}$ no communication point .

Remark that only the set V_1 is solid while the other four sets have empty interior. It means that if a feasible control of problem P exists, then it will probably be an alternate communication.

The above defined sets, $V_i, i = 1,\dots,5$, depend only on the values of λ_1, λ_2 and t_1, t_2. This means that given the coefficients of matrix Q, i.e. saturation aversion, word-of-mouth productivity, goodwill decay and goodwill productivity, we can **understand the structure (the type) of the optimal communication.**

2.6 Discussion

The communication expenditure minimization model considered is rather general and takes into consideration different generalizations of the classical Nerlove and Arrow model which can be found in literature. Nevertheless we did not address some typical marketing issues [8].

Our model responds dynamically to increases of communication expenditure but the effect of promotion on sales (γ_x) is constant and does not depend on sales level.

Linearity is a strong assumption but we consider seasonal goods and the selling period is short. The problem we have faced might be considered an acceptable linear approximation of more realistic models that consider concave or S-shaped advertising return functions.

To consider only constant parameters is a shortcoming of the model too; for example we consider the goodwill effect on sales (i.e. the parameter ϵ_A) as fixed but if it would depend on time and on communication efforts (see e.g Naik, Mantrala and Sawyer [17]) this would also allow us to take into account *wearout* effects, and this way the problem would become non-linear.

Nevertheless the main meaning of the proposed model relies on its simple qualitative throughput and on putting together different promotional activities like advertising, sales promotion and the effect of word-of-mouth.

We have seen in particular that the sign of $\theta\delta_A - \varphi\delta_x$ determines the type of optimal alternate communication. This implies that if saturation aversion and goodwill decay, which can be considered as negative factors for the firm, are "stronger" than word-of-mouth and goodwill productivity, which are positive for the firm, then it is more convenient to advertise only in the middle of the selling period (see (3)). Otherwise it is convenient to advertise at the beginning of the selling period and then to refresh the goodwill of the firm at the end of the period (see (2)).

The model considered can be generalized in different ways. In [4] we consider a firm which produces and sells a product in two consecutive time intervals and propose an optimal control model for its marketing. The production period state variables are the inventory level and two goodwills (one for the consumer and one for the retailer) while the selling period state variables are the sales level and the two goodwills. In the production period there are three controls: on production, quality and advertising, while in the selling period the only control allowed is on communication (cf. with the model considered

[8] For a summary of the properties an advertising response model should have see e.g. [15].

in this work). The linear case is considered in detail and transformed into an equivalent non linear programming problem. Moreover, in [2] we consider a linear optimal control model for the marketing of products which are produced by the same firm and sold by retailers in different market **segments**.

3 Trade discount policy in a distribution channel

In this section we discuss some results reported in [5], [6], [7], where we considered a vertical control distribution channel in which a manufacturer sells a single kind of good to a retailer. The retailer then sells the good to the consumers.

To earn a reasonable profit the members of a distribution channel often adopt rather simple pricing techniques. For example, manufacturers may use cost-plus pricing, simply defining the price adding a desired profit margin to (variable) production costs; in a similar fashion, retailers very often use to determine shelf prices adding a fixed percentage markup to the wholesale price.

The main advantage of simple policies is that they are...easy to be applied. But this blind approach to pricing does not provide tools to manufacturers in order to encourage retailers to sell and retailers, in turn, cannot adequately stimulate consumer to buy.

We will focus on the effects of trade promotions, a widely used dynamic pricing strategy that manufacturers can exploit to raise sales. With trade promotions an incentive mechanism is used to drive other channel members' behaviors.

In particular we investigate the relationships between the members of a distribution channel by means of optimal control models in a stylized vertical distribution channel: a manufacturer serves a single segment market through a single retailer and a contract fixes a trade discount policy which will be followed by the contractors.

Trade discounts have usually a double positive effect on sales since part of the wholesale price reduction may be transferred to the shelf price (pass-through) and part of the discount will be kept by the retailer who will be more motivated [15],[16], and higher motivation means higher effort in selling the product.

3.1 Two optimal control models

Our starting point is given by a couple of models presented in [5] and [6] where we considered a stylized vertical channel in which a manufacturer sells a single product during the limited time period $[t_1, t_2]$. In those models the manufacturer sells to a single retailer, her aim is to maximize the total profit in the given time period. By means of trade discounts the manufacturer can raise her sales both because the retailer transfers a part of the discount to the shelf price (pass-through) and because if the retailer will keep part of the incentive for himself he will be more motivated in selling the specific product, thus giving another upward push to sales.

This way two optimal control models can be considered in which the controls are, respectively, trade discount (the manufacturer's control) and pass-through (the retailer's control). The state variables in the models are the cumulative sales and the retailer's motivation.

Let us define the details of the models. Define

$$\begin{aligned} x(t) &= \text{cumulative sales during the time period } [t_1, t], \\ p_w(t) &= \text{wholesale price at time } t, \\ c_0 &= \text{unit production cost}, \\ \alpha(t) &= \text{trade discount at time } t,\ \alpha(t) \in [\alpha_1, \alpha_2] \subseteq [0,\ 1], \\ \beta(t) &= \text{pass-through at time } t,\ \beta(t) \in [\beta_1, \beta_2] \subseteq [0,\ 1]. \end{aligned}$$

Constants α_1, α_2, β_1, β_2 represent the boundary values of trade discount and pass-through that manufacturer and retailer require not to be exceeded in order to participate in the selling activity of the channel. In particular manufacturer establishes the values of α_2 and β_1 while the retailer fixes the values of α_1 and β_2.

Considering the trade discount explicitly, the wholesale price can be rewritten as $p_w(t) = p(1-\alpha(t))$ where p is the wholesale price when no trade discount is applied.

Remark that $\dot{x}(t)$ represents the sales rate at time t; we suppose that it coincides with the consumer's demand at time t and that the firm will produce exactly the quantity to be sold.

The total profit of the manufacturer can be written as

$$\int_{t_1}^{t_2} (p_w(t) - c_0)\dot{x}(t)dt,$$

or, since $x(t_1) = 0$,

$$J_M = qx(t_2) - p \int_{t_1}^{t_2} \dot{x}(t)\alpha(t)dt, \tag{14}$$

where $q = p - c_0$. In order to obtain a non negative profit the manufacturer will ask $\alpha_2 \leq q/p$ as it is shown in [5].

The total profit of the retailer is then

$$J_R = p \int_{t_1}^{t_2} \alpha(t)(1 - \beta(t))\dot{x}(t)dt. \tag{15}$$

If the retailer's sales motivation at time t, summarized by the state variable $M(t)$, is increasing with respect to consumer's demand and to trade discount then its dynamics can be described by

$$\dot{M}(t) = \gamma\dot{x}(t) + \varepsilon(\alpha(t) - \overline{\alpha}),$$

where γ and ε are strictly positive constants. Constant $\overline{\alpha} \in (\alpha_1, \alpha_2)$ takes into account the fact that the retailer has some expectations on trade prices, his motivation decreases if his expectations are disappointed, i.e. if $\alpha(t) < \overline{\alpha}$, and increase if $\alpha(t) \geq \overline{\alpha}$.

The dynamics of the total amount of sales at time t is given by

$$\dot{x}(t) = -\theta x(t) + \delta M(t) + \eta\beta(t)\alpha(t),$$

where δ, η, and θ are strictly positive. Constant δ represents the retailer's selling skill while θ is needed to model the market saturation effect (e.g. large markets will display low values of θ).

The manufacturer's profit maximization problem leads then to the following optimal control problem (see [5])

$$\begin{array}{lll} M: & \text{maximize} & J_M \\ & \text{subject to} & \dot{x}(t) = -\theta x(t) + \delta M(t) + \eta\beta(t)\alpha(t), \\ & & \dot{M}(t) = \gamma\dot{x}(t) + \varepsilon(\alpha(t) - \overline{\alpha}), \\ & & x(t_1) = 0, \quad M(t_1) = \overline{M}, \\ & & \alpha(t) \in [\alpha_1, \alpha_2] \subseteq [0,\ 1], \\ & & \beta(t) \in [\beta_1, \beta_2] \subseteq [0,\ 1], \end{array}$$

where $\overline{M}$ is the initial motivation of the retailer (we assume $\overline{M} > 0$).

For the case of constant $\beta(t)$ problem M has been investigated in [5], [6]. In particular, it was shown that the structure of the optimal price discount strategy over time has two essentially different forms depending on the retailer's efficiency, i.e. is selling skill, aptitude,..., which is described by the model parameter δ. More precisely if δ is less than a given threshold, which means that the retailer is a rather bad seller, the optimal policy is to progressively increase the discount during the selling period. If the retailer is a good seller instead, i.e. δ lies above the threshold, the best policy is first to increase the discount and then to decrease it. This means that a good retailer, if properly motivated by initial discounts, can provide a good sales level also when discounts become lower. With bad retailers it is always necessary to increase the discount to obtain a higher profit reached through a price policy since "service" is not good enough.

In a similar way it is possible to formulate (see [6]) a corresponding retailer's optimal control problem R, keeping the same motion equations and constraints and with objective functional J_R.

3.2 Trade discount policies in the differential games framework

In [6] and [7] we addressed some preliminary considerations on the differential game, which will be denoted by MR, defined by the objective functionals

$$J_M \ , \quad J_R \ ,$$

by the motion equations

$$\begin{aligned}
&\dot{x}(t) = -\theta x(t) + \delta M(t) + \eta\beta(t)\alpha(t), \\
&\dot{M}(t) = \gamma\dot{x}(t) + \varepsilon(\alpha(t) - \overline{\alpha}), \\
&t \in [t_1, t_2], \quad x(t_1) = 0, \quad M(t_1) = \overline{M},
\end{aligned}$$

and by the constraints

$$\begin{aligned}
&\alpha(t) \in [\alpha_1, \alpha_2] \subseteq [0,\ 1], \\
&\beta(t) \in [\beta_1, \beta_2] \subseteq [0,\ 1].
\end{aligned}$$

As a first step to study the game MR in [6] and [7] we considered the simplified framework in which both controls must take a constant value in the whole time period $[t_1, t_2]$ and these values are decided at time t_1. In this case the solution of problems M and R becomes straightforward and allows to obtain some properties of the differential game MR.

With constant controls $\alpha(t) = \alpha$ and $\beta(t) = \beta$ the manufacturer's and retailer's profits are functions of α and β :

$$J_M = J_M(\alpha\ ,\ \beta)\ , \quad J_R = J_R(\alpha\ ,\ \beta).$$

3.3 Nash equilibria

Let us look now for the Nash equilibria of the differential game MR. Functional J_M is concave with respect to α, while functional J_R is concave with respect to β (see [7]). Nash equilibria are therefore the solutions of the system

$$\begin{cases} \dfrac{\partial J_M}{\partial \alpha} = 0 \\ \\ \dfrac{\partial J_R}{\partial \beta} = 0 \end{cases}$$

In [7] we found the necessary and sufficient conditions under which Nash equilibria exist and, moreover, are feasible, i.e. belong to $[\alpha_1\ ,\ \alpha_2] \times [\beta_1\ ,\ \beta_2]$. Further, we compared the manufacturer's and the retailer's profits in the different Nash equilibria points. In particular, we found the Nash equilibria point which is the best for both manufacturer and retailer.

3.4 Stackelberg equilibrium

A different point of view on the channel marketing activity can be obtained considering the manufacturer and the retailer as the two players of a Stackelberg game (see [8] and [12]).

We first consider **the manufacturer as the channel leader:** in this case we assume that she can only choose a constant trade discount during the whole sales period. This way we formulate the following Stackelberg game:

$$ML: \text{ maximize } \ qx(t_2) - p\alpha \int_{t_1}^{t_2} \dot{x}(t)\,\mathrm{d}t, \quad \alpha \in [\alpha_1\ , \alpha_2],$$

where, for each fixed α, functions $x(t)$, $M(t)$, $\beta(t)$ are optimal solution of

$$\begin{aligned} &\text{maximize} && p\alpha \int_{t_1}^{t_2} \dot{x}(t)(1-\beta(t))\,\mathrm{d}t, \\ &\text{subject to} && \dot{x}(t) = -\theta x(t) + \delta M(t) + \eta\alpha\beta(t), \\ &&& \dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha - \overline{\alpha}), \\ &&& x(t_1) = 0, \quad M(t_1) = \overline{M}, \quad \beta \in [\beta_1\ , \beta_2]. \end{aligned}$$

When both controls are constant, we can rewrite the Stackelberg game this way:

$$ML: \text{ maximize } \; J_M(\alpha,\beta) = (q - p\alpha)x(t_2), \quad \alpha \in [\alpha_1 \, , \alpha_2],$$

where, for each fixed α, β is the optimal solution of

$$\text{maximize } \; J_R(\alpha,\beta) = p\alpha(1-\beta)x(t_2) \; , \beta \in [\beta_1 \, , \beta_2].$$

Consider now **the retailer as the channel leader:** we assume in this case that he can only choose a constant pass-through during the whole sales period. This way a new Stackelberg game can be formulated as follows:

$$RL: \text{ maximize } \; (1-\beta)p \int_{t_1}^{t_2} \dot{x}(t)\alpha(t)\,\mathrm{d}t, \quad \beta \in [\beta_1 \, , \beta_2],$$

where, for each fixed β, functions $x(t)$, $M(t)$ and $\alpha(t)$ are optimal solution of

$$\begin{array}{ll} \text{maximize} & qx(t_2) - p\displaystyle\int_{t_1}^{t_2} \dot{x}(t)\alpha(t)\,\mathrm{d}t, \\ \text{subject to} & \dot{x}(t) = -\theta x(t) + \delta M(t) + \eta\beta\alpha(t), \\ & \dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha(t) - \overline{\alpha}), \\ & x(t_1) = 0, \quad M(t_1) = \overline{M}, \quad \alpha(t) \in [\alpha_1, \alpha_2]. \end{array}$$

When both controls are constant, the Stackelberg game can be formulated as follows:

$$RL: \text{ maximize } \; J_R(\alpha,\beta) = p\alpha(1-\beta)x(t_2) \; , \; \beta \in [\beta_1 \, , \beta_2],$$

where, for each fixed β, α is the optimal solution of

$$\text{maximize} \qquad J_M(\alpha,\beta) = (q - p\alpha)x(t_2) \; , \quad \alpha(t) \in [\alpha_1, \alpha_2].$$

In [7] we found the necessary and sufficient conditions under which the two types of Stackelberg equilibria exist and, moreover, are feasible, i.e. belong to $[\alpha_1 \, , \, \alpha_2] \times [\beta_1 \, , \, \beta_2]$. Further, we compared the manufacturer's and the retailer's profits in the two types of Stackelberg equilibria points. In particular, it was found **an important role** of the following parameter:

$$C = \frac{pK}{q(H+L)} \; ,$$

where

$$H = \frac{\eta}{a}\left(1 - e^T\right) > 0, \quad L = -\frac{\delta\epsilon}{a^2}\left(1 - e^T + T\right) > 0,$$
$$K = \frac{\delta}{a^2}\left[\left(a\overline{M} + \epsilon\overline{\alpha}\right)\left(1 - e^T\right) + \epsilon\overline{\alpha}T\right] = \frac{\delta}{\epsilon}\ \overline{M}H - \overline{\alpha}L$$

and $a = \theta - \gamma\delta,\ T = a(t_1 - t_2)$ [(9)].

Further, let $(\alpha^{ML},\ \beta^{ML})$ be Stackelberg equilibria point when manufacturer is leader, while $(\alpha^{RL},\ \beta^{RL})$ be Stackelberg equilibria point when retailer is leader. Let us denote

$$J_M^{ML} = J_M(\alpha^{ML},\ \beta^{ML}), \qquad J_R^{ML} = J_R(\alpha^{ML},\ \beta^{ML}),$$

$$J_M^{RL} = J_M(\alpha^{RL},\ \beta^{RL}), \qquad J_R^{RL} = J_R(\alpha^{RL},\ \beta^{RL}).$$

The role of parameter C is the following: it is possible to find $C_1 < C_2$ such that:

1) if $C < 0$ or $C > C_1$, then $J_M^{ML} > J_M^{RL}$ and $J_R^{RL} > J_R^{ML}$, i.e. both manufacturer and retailer want to be leader;
(CONFLICT SITUATION: **the struggle to be leader**)

2) if $C \in (0, C_1)$ then $J_M^{ML} < J_M^{RL}$ and $J_R^{RL} < J_R^{ML}$, i.e. both manufacturer and retailer want to be follower;
(CONFLICT SITUATION: **the struggle to be follower**)

3) if $C \in (C_1\ , C_2)$ then $J_M^{ML} < J_M^{RL}$ and $J_R^{RL} > J_R^{ML}$, i.e. manufacturer wants to be follower, while retailer wants to be leader.
(**THE SITUATION OF AGREEMENT!!!**)

Moreover, it was found that the constants C_1 and C_2 **do not depend from the parameters of the models!** To clarify the economical meaning of the constants may be a theme of future research.

Another theme of the future research may be to consider the case when feasible controls (trade discount $\alpha(t)$ and pass-through $\beta(t)$) are not constant, but piece-wise constant. Moreover, it can be interesting to consider not only the selling period, but also the production one. Finally, we could assume that the market is not homogeneous, but segmented (cf. [2]).

[9)] We assume, due to economical reasons, that $a > 0$.

References

[1] A. Buratto and D. Favaretto, Optimal communication mix to maximize brand image. *Rapporto n.95/2001, Dipartimento di Matematica Applicata, Università di Venezia* (2001).

[2] I. Bykadorov, A. Ellero and E. Moretti,A linear optimal control model for multi-segment marketing. *Rapporto n.100/2001, Dipartimento di Matematica Applicata, Università di Venezia* (2001).

[3] I. Bykadorov, A. Ellero and E. Moretti, Minimization of communication expenditure for seasonal products. *RAIRO Operations Research* **36** (2002) 109–127.

[4] I. Bykadorov, A. Ellero and E. Moretti, A model for the marketing of a seasonal product with different goodwills for consumer and retailer. *Journal of Statistics & Management Systems* **6** (2003) 115–133.

[5] I. Bykadorov, A. Ellero and E. Moretti, Optimal control of retailer's motivation by trade discounts. *Rapporto n.119/2004, Dipartimento di Matematica Applicata, Università di Venezia* (2004).

[6] I. Bykadorov, A. Ellero and E. Moretti, Optimal control of trade discount in a vertical distributional channel. *Rendiconti per gli Studi Economici Quantitativi, 2005, Dipartimento di Matematica Applicata, Università di Venezia* (2005) 121-129.

[7] I. Bykadorov, A. Ellero and E. Moretti, Trade discount policies in the differential games framework: preliminary results. *Rapporto n.130/2005, Dipartimento di Matematica Applicata, Università di Venezia* (2005).

[8] E. Dockner, S. Jœrgensen, Ngo Van Long and G. Sorger, *Differential Games in economics and management science.* Cambridge Un. Press, Cambridge (2000).

[9] D. Favaretto and B. Viscolani, A single production and advertising control problem with bounded final goodwill. *Journal of Information and Optimization Sciences* **21** (2000) 337–357.

[10] G. Feichtinger, R. F. Hartl and S. P. Sethi, Dynamic optimal control models in advertising: recent developments. *Management Science* **40** (1994) 195–226.

[11] S. Funari and B. Viscolani, Advertising and congestion management for a museum temporary exhibition. *Central European Journal of Operations Research* **10** (2002) 149–161.

[12] S. Jœrgensen, G. Zaccour, *Differential games in marketing.* Kluwer Ac. Pub., London (2004).

[13] E. E. Lawler, *Motivation in work organizations.* Brooks/Cole, Monterey - Cal (1973).

[14] G. L. Lilien, P. Kotler and K. S. Moorthy, *Marketing models.* Prentice Hall Int., Englewood Cliffs (1992).

[15] J. D. C. Little, Aggregate advertising models: the state of the art. *Operations Research* **27** (1979) 629–667.

[16] T.R. Mitchell, Motivation: new directions for theory, research and practice. *Academy of management review* **7** (1982) 80–88.

[17] P. A. Naik, M. K. Mantrala and A. G. Sawyer, Planning media schedules in the presence of dynamic advertising quality. *Marketing Science* **17** (1998) 214–235.

[18] M. Nerlove and K. J. Arrow, Optimal advertising policy under dynamic conditions. *Economica* **29** (1962) 129–142.

[19] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes.* Pergamon Press, London et al. (1964).

[20] A. Seierstad and K. Sydsæter, *Optimal Control Theory with Economic Applications.* North–Holland, Amsterdam (1987).

[21] K. Spremann, The signaling of quality by reputation, in *Optimal Control Theory and Economic Analysis 2,* edited by G. Feichtinger. North–Holland, Amsterdam (1985) 235–252.

[22] M. L. Vidale and H. B. Wolfe, An operations research study for sales response to advertising. *Operations Research* **5** (1957) 370–381.