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Constrained matchings in bipartite graphs *

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Abstract

We consider some particular cases of the general problem which consists of stating if a shift vector of m elements corresponds to a permutation of the same number of objects. Shifts are defined as the steps each element in a given permutation must perform in order to reach its natural position, i.e., to its position in the fundamental permutation, (1, 2, ..., m). The problem, never studied at our knowledge, was suggested to us by a paper in which a certain number of vehicles must be moved from initial to final positions in a grid graph in such a way that vehicles do not share the same vertex at the same time. The problem can also be stated in terms of a covering one, i.e. the vertex cover of a particular graph, with cycle using arcs of defined lengths. Some properties of the shift vector, necessary in order to guarantee the existence of a corresponding permutation, and a general algorithm to find the permutations, if they exist, are given.

JEL classification: C61, C63
Keywords: Cycle covering, Permutations

1 Introduction: the problem

Let $G = (U, V; E)$ be a complete bipartite graph in which $U = \{x_1, x_2, \ldots, x_m\}$ and $V = \{y_1, y_2, \ldots, y_m\}$ are the nodes sets (note $|U| = |V| = m$) and $E = \{e_{ij} = (x_i, y_j)\}$ is the set of the edges. Associated with each edge e_{ij} is a weight, its length, defined as follows

$$l(e_{ij}) = l_{ij} = (j - i).$$

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A matching M in G is a set of m disjoint edges, neither of them having nodes in common. It is a well-known fact that every matching in such a graph corresponds to a permutation of the elements of the set $\{1, 2, \ldots, m\}$: we can think of this permutation dynamically, as a set of new positions elements in U get travelling into V. In this way, given a matching

$$M = \{e_{1j_1}, e_{2j_2}, \ldots, e_{mj_m}\},$$

the corresponding permutation can be described as

$$P = (j_1, j_2, \ldots, j_m).$$

The lengths of the edges in M, respectively, $(j_1 - 1), (j_2 - 2), \ldots, (j_m - m)$, denote the number of steps every element $y_i \in V$ must perform to come back to its "natural" position in the fundamental permutation $(1, 2, \ldots, m)$. This number, which will be called "shift", is positive, if the corresponding element in P must move toward right; negative if it moves to the left. Obviously, it can be zero. The set of shifts (lengths) will be denoted like a vector, v. In this way, we have

$$v = (l_{1j_1}, l_{2j_2}, \ldots, l_{mj_m}),$$

and the i^{th} element is $l_{ij_i} = j_i - i$. It is easy to verify that the sum of the elements in v is 0: in this way if the elements in v are not all zero, they must be in part positive ones, in part negative. We must observe that to each permutation corresponds a vector v (no two permutations share the same v); on the contrary, a generic vector v of m components, all satisfying $-(m-1) \leq v_i \leq (m-1)$, in which the sum of the components is zero, does not necessarily correspond to a permutation. Now, re-order elements in v, from the greatest one to the lesser one. We shall call the vector of such elements w. We ask if, for an assigned set of "ordered lengths" (shifts)

$$w = (w_1, w_2, \ldots, w_m)$$

with

$$w_1 \geq w_2 \geq \ldots \geq w_m$$

and

$$\sum_{i=1}^{m} w_i = 0,$$

there exists a permutation having like its length vector a rearrangement of these values w_i. This will be called *Problem A*. Besides the intrinsic theoretical interest from the point of view of permutation combinatorics (i.e. to state under which hypothesis a vector w can induce a permutation on integers 1 to m), this problem arose in an environment of routing a fleet of vehicles on grids, analyzed in [1].

Suppose we have a grid-graph with n rows and m columns. Columns correspond to vertical lanes. Each row corresponds to a horizontal lane and is also called level: the first level is the lowest one. A certain number of vehicles, at most m vehicles, in $t = 0$ is located on nodes of the first level. Each of them must move to the opposite side (the top) of the grid (every two different vehicles must reach different positions). During a time unit every vehicle can move from a node to an adjacent one in the grid: the presence of two vehicles at the same time in the same node of the grid is forbidden. In [1] Authors shows how to perform the trip of each vehicle in such a way that the total time spent by vehicles to reach their destination is minimised. The path depends exclusively on the horizontal distance from the initial position of the vehicle to its destination: vehicles whose horizontal distance is larger must be moved first. So, in order to state how many steps are needed to reach destination points, only the structure of the vector w above introduced is relevant.

If we add some other constraints to the *Problem A*, the model can be suited for other problems of practical interest. In particular, if the components of w are constrained to assume few values, we can ask for the possibility to move every element in U to another position in V in such a way that the distance origin-destination be in a limited range of prefixed values: for example a public official could be chosen to be living in a town different from the one in which he works, but not so much far away for travel purposes (problem B). In this paper we analyze some particular cases of vectors w in order to give some useful rules when studying if a permutation having these values as shifts length exists. Another well known issue regarding bipartite matching, is the correspondence of each matching in the bipartite graph G with a set of (sub)circuits covering all vertices in a complete graph $G^* = (N, A)$ in which the set of nodes can be denoted as $N = 1, 2, \ldots, m$ and the (directed) arc (i, j) has length $(j - i)$. G^* is comprehensive of loops. Vector w is a set of lengths (possibly, with repeated values) and what we want is to find a covering of G^* which uses exactly the lengths in w. An obvious representation of G^* is the one obtained considering the m nodes as integer coordinate points on an (horizontal) axis, and the arcs have positive length, if directed in agreement with the orientation of the axis; they have negative weight in the other case (loops have length 0). In this way the problem can be viewed in two different ways: as a geometric one considering this last representation; a problem in permutation combinatorics viewed from an algebraic point of view. The paper is organised as follows. In the first section we first introduce some other relevant definitions and notations, particularly the shift table, then we formulate problem A as the search for a solution in a system of integer linear equalities. In section 2 some relevant properties of vector v and w necessary in order to give raise to a permutation, are given. In section 3 we analyse classes of vector w for which we can easily state the existence or not of a corresponding permutation. In section 4 we present an algorithm, similar to the Robert-Flores technique to find Hamiltonian circuits in graphs, which can solve Problem A at least as far as we have some tenth of nodes (elements). For what it concerns the complexity, it is nowadays an open question. It is a challenge for future researches, as well as more general rules capable of handling other classes of w vectors.

2 Definitions and notations

Given a permutation

$$P = (j_1, j_2, \ldots, j_m)$$

of the integers $1, 2, \ldots, m$, the **shift** of the element j_k is the quantity

$$v_k = j_k - k.$$

The vector of the shift of the elements in P will be denoted as $v(P)$. In this way we have

$$v(P) = ((j_1 - 1) \quad (j_2 - 2) \quad \ldots \quad (j_m - m)).$$

As seen before, shifts represent the number of steps each element in P must perform in order to go into its own position in the fundamental permutation $(1, 2, \ldots, m)$: steps are positive, if the element must be moved toward the right side, otherwise they are negative [2]. We shall denote with $w(P)$ or simply, for sake of brevity, w, the vector of ordered shifts, i.e. the shifts rearranged from the greatest to the smallest one. In this way, by definition, we have:

$$w_1 \geq w_2 \geq \ldots \geq w_m.$$

In a cartesian system, consider the m^2 points (x, y) with integer coordinates such that $1 \leq x \leq m; 1 \leq y \leq m$. To each point (i, j) of these we associate the value (weight) $d(i, j) = j - i$. We shall call **grid** D this set of points with associated weights. The elements in D can be thought as elements of a matrix: the only difference lies in the numbering of rows (also called levels) and columns. In D the first row is the one of the elements with $y = 1$; obviously, the first column is the one which corresponds to $x = 1$.

In D the elements such that $x = y$ (primary diagonal) all have a weight 0. The elements with the same weight k give raise to the diagonal $D(k)$.

A generic permutation $P = (j_1, j_2, \ldots, j_m)$ corresponds in D to a set of m elements, i.e. the elements $(x = i, y = j_i)$, one for each row and one for each column. Moreover, the weights of the elements in D corresponding to a permutation are their shifts, as defined above.

With reference to the grid D, **Problem A** can be stated as it follows:
given a vector

$$w = (w_1 \quad w_2 \quad \ldots \quad w_m)$$

such that

$$w_1 \geq w_2 \geq \ldots \geq w_m, \quad -(m-1) \leq w_i \leq (m-1), \quad \sum w_i = 0,$$

find m elements in D, one for each row and one for each column, such that in the diagonal $D(k)$ there are as many elements as many times the value k is replicated in w. If this is possible (i.e., if a permutation having w has vector of its rearranged shifts exists), we call w **admissible**.

In this way, as in general values in w are not necessarily distinct, suppose we have

$$w_1 = w_2 = \ldots = w_{m_1} = k_1,$$

$$w_{m_1+1} = w_{m_1+2} = \ldots = w_{m_1+m_2} = k_2,$$

$$w_{m_1+m_2+1} = w_{m_1+m_2+2} = \ldots = w_{m_1+m_2+m_3} = k_3, \ldots$$

then the permutation we search must have (exactly) m_1 elements in diagonal $D(k_1)$, m_2 elements in diagonal $D(k_2)$, m_3 elements in diagonal $D(k_3)$ and so on. For sake of brevity, we denote such a vector as

$$w \quad ((k_1)_{m_1} \quad (k_2)_{m_2} \quad \dots \quad (k_v)_{m_v}).$$

From another point of view, Problem A can be formulated as the one which consists of solving an integer linear equation system. Given a vector w, of m components, let x_{ij} be a Boolean variable which assume value 1 if and only if a permutation of the elements $(1, 2, \dots, m)$ has element j in position i. Let us denote m_k the number of times the value k is contained in w. If vector w corresponds to a permutation, the following system must have integer solution(s):

$$\sum_{i=1}^{m} x_{ij} = 1, \ \forall j, \tag{1}$$

$$\sum_{j=1}^{m} x_{ij} = 1, \ \forall i, \tag{2}$$

$$\sum_{i=1}^{m-k} x_{i,i+k} = m_k, \ k = 0, 1, \dots m, \tag{3}$$

$$\sum_{i=k+1}^{m} x_{i,i-k} = m_{-k}, \ k = 1, 2, \dots m, \tag{4}$$

$$x_{ij} \in \{0, 1\}, \ \forall i, j. \tag{5}$$

Here, the value m_k denotes how many times the element k is replicated in w. The four sets of equations are, respectively, the usual condition on bipartite matching (equations (1) and (2)); equations (3) and (4) impose the number of elements that the matching must take for each of the diagonals of D. Obviously, in the above formulation one can erase all the variables which have necessarily value 0 (and all the equations which contain only this kind of unknown).

3 Some properties of vectors v and w

Vector $v = v(P) = (v_1 \quad v_2 \quad \dots \quad v_m)$ enjoys two properties:

(3.1) $\sum_{i=1}^{m} v_i = 0$;

(3.2) $v_{i+k} \neq v_i - k, \ \forall k \in \{-i+1, -i+2, \dots, m-i-1, m-i\}$.

The first one is quite immediate. By its own definition, we have:

$$\sum v_i = \sum (x_i - i) = \sum x_i - \sum i = 0$$

(observe: the two last summations are both the sum of the first m integer numbers).

To prove the second property, observe that the element v_i belongs to the i-th column and row $j = v_i + i$ in D. m elements in the grid D correspond to each permutation P, each on a different row and column. So row j cannot contain other elements of P. Elements in j have values (shifts) $v_i + i - 1$, (in the first column); $v_i + i - 2$, (in the second column),..., $v_i + i - m$ (in the last one). The corresponding cells of D are all forbidden and this conclude the proof. In order vector w be admissible, it must satisfy the following (necessary) conditions:

(3.3) $\sum_{i=1}^{m} w_i = 0$;

(3.4) $\sum_{i=1}^{h} w_i \leq h(m-h)$.

The first condition is an obvious consequence of the same condition on vector v. The second condition gives some upper bounds on the sum of the first highest h elements in w. To prove it, observe that it is true when $h = 1$: the element of the grid of maximum value is $m - 1$. By direct inspection, it is easy to control the same property when $h = 2$ or 3. In general, we must observe that the value of the elements in the grid D is decreasing when moving from the diagonal $D(m-1)$ (which consists of one single element) down toward the principal diagonal (when value is 0) and so on toward the minimal element $-(m-1)$. The h major values in w are contained in the North Western diagonals. Partition now D in four blocks:

$$D = \begin{matrix} H & Q \\ R & S \end{matrix}$$

with H a sub-grid hXh. Consequently, S is $(m-h)X(m-h)$. We can distinguish two cases: the h highest values in w are all contained in H or not. In the first case, the sum of the h values is exactly $h(m-h)$, independently of the rows and columns in which these h highest elements of w are located. In the second case, generally speaking, we have $j < h$ highest elements in H (possibly, $j = 0$) and the other $(h-j)$ ones partitioned between R, S, Q. Now, erase the rows and columns of D corresponding to this set of j highest elements in H. Call H^*, R^*, Q^* the resulting blocks (S remains unchanged). The $h - j$ highest elements, all outside H, can be compared with the diagonal elements in H^*. It is easy to establish a $1-1$ dominance relationship between the diagonal elements in H and the other $h - j$ highest values in w and so the proof can be concluded.

4 About the admissibility of vectors $w \quad (k_{m/2} \quad -k_{m/2})$

In this section we analyse vectors w in which components may take only two values, i.e. k and $-k$. Obviously, the number of elements in w must be even.

Proposition 1 A necessary and sufficient condition in order $w \quad (k_{m/2} \quad -k_{m/2})$ (m even) be admissible, is that $m/(2k)$ be an integer number.

Proof

a) The condition is sufficient: if $m/(2k)$ is integer, then $w \quad (k_{m/2} \quad -k_{m/2})$ is admissible. In fact, consider the simplest case in which $m = 2k$. In this case, the elements in the two diagonals $D(k)$ and $D(-k)$ are distributed one for each row and one for each column of D: so we get a permutation. If m is any integer multiple of $2k$, we can partition D in $(m/(2k))^2$ blocks, each of them having dimension $2kX2k$. We get a permutation P by choosing in every block along the principal diagonal the elements equal to k and the ones equal $-k$.

b) The condition is necessary: if m is not an integer multiple of $(2k)$, a permutation corresponding to $w \ \ (k_{m/2} \ \ -k_{m/2})$ does not exist. Suppose $m = (2k)j + m$, with $0 < m < 2k$. We can partition D in $(j+1)^2$ blocks as it follows: j^2 blocks B of size $(2kX2k)$ surrounded from: j blocks B^1 $(mX2k)$ at the top; j blocks B^2 $(2kXm)$ to the right; a block C (mXm) at the top of the principal diagonal of D. Because in building the permutation P we must use only elements of value k or $-k$, we see that: the first k column must be covered with elements of value k, taken form $D(k)$; the first k rows must be covered form elements of value $-k$ (taken in $D(-k)$); this accommodate for the first (lowest) diagonal block; for each of the remaining diagonal blocks $(2kX2k)$ we repeat this operation. The last block C cannot be covered with elements of value k or $-k$, because they lacks, when $m \leq k$; on the other hand, when $k < m < 2k$, they cannot cover the k^{th} row and the k^{th} column in C. So the permutation, we are searching for, does not exist.

5 About the admissibility of vectors $w \ \ ((k+1)_a \ \ k_b \ \ -k_c \ \ -(k+1)_d)$

When m is even we cannot have a permutation with two shifts alone, k and $-k$: the shifts must take at least two different (absolute) values. So, generally speaking, in order to have shifts in such a way that they be the least different one from another as it is possible, we must perform our analysis for the following cases:
$C_1: \ \ w \ \ ((k+1)_a \ \ -k_c)$ (or, equivalently, $w \ \ (k_b \ \ -(k+1)_d)$;
$C_2: \ \ w \ \ ((k+1)_a \ \ k_b \ \ -(k+1)_d)$;
$C_3: \ \ w \ \ ((k+1)_a \ \ k_b \ \ -k_c)$;
$C_4: \ \ w \ \ ((k+1)_a \ \ k_b \ \ -k_c \ \ -(k+1)_d)$.
Let us begin with class C_1.
Observe, first of all, that in order w be admissible, as a consequence of the property (3.4):
- when only two (absolute) values are considered, and m is even, the largest one cannot be greater than $m/2$;
- when m is odd, the vector w with the largest (absolute) entries is

$$w \ \ (((m+1)/2)_{(m-1)/2} \ \ -((m-1)/2)_{(m+1)/2})$$

which is always admissible, whatever be m. To complete our analysis, with regard to case C_1, one can easily see (i.e., solving a Diophantine equation) that all the vectors w belonging to this set can be obtained letting $a = hk$ and $b = h(k+1)$ (obviously, as a consequence, it must be $m = h(2k+1)$). They are all admissible, because a permutation is obtained using in the grid D, starting from bottom left, alternately, k consecutive elements on the diagonal $D(k+1)$; then $k+1$ consecutive elements of $D(-k)$ and so on repeating this h times.

Even vectors w belonging to the classes C_2, C_3 and C_4 which have the properties (3.3) and (3.4) (section 3) can be easily obtained by solving a suitable system of diophantine equations, (by example, in the case C_4, $a(k+1) + bk = ck + d(k+1)$, with the constraint $a + b + c + d = m$). The results we can establish for these three classes will be formulated first of all in the case of the class C_4 and then extended to the other ones.

Proposition 1 *The inner exchange rule*
When m is even, vectors

$$w \quad ((m/2)_{(m/2)-1} \;\; [(m/2)-1]_1 \quad -[(m/2)-1]_1 \quad -(m/2)_{(m/2)-1})$$

and

$$w \quad ((m/2)_{(m/2)-2} \;\; [(m/2)-1]_2 \quad -[(m/2)-1]_2 \quad -(m/2)_{(m/2)-2})$$

are always admissible.
Proof
Vectors $w \quad ((m/2)_{(m/2)} \quad -(m/2)_{(m/2)})$ are admissible (see analysis in section 4). In the representation of the permutation by elements on the grid D, these vectors correspond to the elements in the two diagonals $D(m/2)$ and $D(-m/2)$. Now, let us substitute in the top row of D the element $m/2$ with $(m/2)-1$ and in the bottom row $-m/2$ with $-((m/2)-1)$. Each column in the new setting contains only one element and so we obtain once again a permutation: it corresponds to the vector

$$w \quad ((m/2)_{(m/2)-1} \;\; [(m/2)-1]_1 \quad -[(m/2)-1]_1 \quad -(m/2)_{(m/2)-1}).$$

Starting with this vector, we can perform another exchange between elements $m/2$ and $m/2-1$ in the first column and symmetrically (between $-m/2$ and $-((m/2)-1)$) in the last column: in this way we obtain another permutation which corresponds to the vector

$$w \quad ((m/2)_{(m/2)-2} \;\; [(m/2)-1]_2 \quad -[(m/2)-1]_2 \quad -(m/2)_{(m/2)-2}).$$

Extensions: generalization of the inner exchange rule
When $m = 2kh$, the above result can be extended as it follows. All vectors

$$w \quad (k_{(m/2h)-j} \;\; (k-1)_j \quad -(k-1)_j \quad -k_{(m/2h)-j}),$$

with $1 \leq j \leq 2h$, are admissible (the above result corresponds to the case in which $h = 1$).
Proof
It is sufficient to apply the procedure in the proof of Proposition 1 to the h diagonal blocks resulting form a partition of D in h^2 sub-grids, all of them of dimension $2kX2k$. Every diagonal block provides a covering of k rows and k columns, using elements in $D(k)$ and $D(-k)$. Then, in every diagonal sub-block we can perform 1 or 2 couple of exchanges $(k \to k-1;\; -k \to -(k-1))$ of the same kind already seen in Proposition 1.

The outer exchange rule
Suppose $m = (2k)h$, k and h integers, $h > 1$. Then, besides the vectors

$$w \quad (k_{m/2} \quad -k_{m/2}),$$

are also admissible the following ones:

$$w \quad ((k+1)_{h-1} \;\; k_{m/2-h+1} \quad -k_{m/2-h+1} \quad -(k+1)_{h-1}).$$

Proof
Following a procedure analogous to the one in the preceding demonstration, now consider diagonal block $(2kX2k)$ whose elements of values k and $-k$ are assumed to belong to the permutation. Exchanges are possible, between adjacent diagonal blocks as it follows. Erase an element k in the top row of the left most block and an element k in the bottom row of the right block, and introduce in the permutation two new elements, $(k+1)$ and $-(k-1)$, which are respectively in the first row above the left block and in the first row below the right block. This procedure can be repeated using the rightmost element $-k$ in the left block and the left most element k in the right block: also these can be substituted by two elements, $k+1$ and $-(k+1)$. The procedure can be performed for every couple of adjacent diagonal blocks, and so in total, at most $(h-1)$ times.
Observation
In section 4, we gave a necessary and sufficient condition in order a vector

$$w \quad (k_{m/2} \quad -k_{m/2})$$

be admissible. A weak condition, which we will use in future in order to exclude sets of w to be admissible, can be based on the concept of column covering. We call a column *covered* if, building a permutation P, we choose an element in the column itself. Obviously, to cover different columns, we must choose elements from different rows of D: as many columns we wish to cover, so as many different rows we must use.

Now, if we must use exclusively elements on the two diagonals $D(k)$ and $D(-k)$, we saw that the first k columns (on the left) can be covered only with elements of value k, taken from the rows from $k+1$ to $2k$ of D; analogously, the last k columns (on the right) must be covered with elements $-k$ taken from rows $m-2k+1$ to $m-k$. A necessary condition in order a permutation P corresponding to $w \quad (k_{m/2} \quad -k_{m/2})$ exists, is that the two following sets be disjoint:

$$I_{left} = \{k+1, k+2, \ldots, 2k\}$$

and

$$I_{right} = \{m-2k+1, m-2k+2, \ldots, m-k\}.$$

I_{left} and I_{right} contain the indices of the rows which cover the first and the last k columns, respectively. This happens if and only if

$$2k \leq (m-2k) \quad or \quad m-k+1 \leq k+1.$$

This condition is equivalent to the following one:

$$k \leq m/4 \quad or \quad k \geq m/2.$$

We can conclude the following result.

Proposition 2
When $m/4 < k < m/2$, there is no permutation P corresponding to the vector $w \quad (k_{m/2} \quad - k_{m/2})$.

For vectors $w \ \ ((k+1)_a \ \ k_b \ \ -k_c \ \ -(k+1)_d)$ a similar proposition can be stated. In this case we must take account that diagonals involved are four: $D(k+1)$, $D(k)$, $D(-k)$ and $D(-k-1)$. It is easy to see that the first k column can be covered by the rows $k+1$ until $2k+1$ (one more row then in the previous case) and in a similar fashion, the k column in the right side of D can be covered only by rows $m-2k$ until $m-k$. The rows involved are, in total, $2k+2$. As a consequence, the two sets:

$$I_{left} = \{k+1, k+2, \ldots, 2k+1\}$$

and

$$I_{right} = \{m-2k, m-2k+2, \ldots, m-k\}$$

can have at most two common elements. This implies that a necessary condition for possible coverings is

$$m-2k \geq 2k \quad or \quad k+1 \geq m-k-1.$$

As ultimate consequence, for values of k such that

$$m/4 < k < (m-2)/2$$

a permutation corresponding to $w \ \ ((k+1)_a \ \ k_b \ \ -k_c \ \ -(k+1)_d)$ does not exist. The other cases are somewhat similar. For vector $w \ \ ((k+1)_a \ \ k_b \ \ -k_c)$ (case C_3) corresponding permutation does not exist when

$$m/4 < k < (m-1)/2;$$

for vector $w \ \ ((k+1)_a \ \ k_b \ \ -(k+1)_d)$ (case C_2) the permutation does not exist provided that

$$(m-2)/4 < k < (m-2)/2.$$

Proposition 3 Queue rule Vectors $w \ \ ((m/2)_a \ \ ((m/2)-1)_b \ \ -((m/2)-1)_c \ \ (m/2)_d)$, in which b and/or $c > 2$, are not admissible.

Proof Let us consider the grid D. We must build a 1-1 correspondence between rows and columns which uses only elements whose weight is $m/2$ or $(m/2)-1$. If in the first column we choose the element $m/2$, this compels us to choose the same elements in all of the following $(m/2)-2$ columns (i.e., until column $(m/2)-1$). This can be seen directly on D, but it is also a consequence of the property 3.2 (section 3). It remains only an element of value $(m/2)-1$ which can be chosen (in column $(m/2)+1$). In turn, now we must choose the element $-((m/2-1)$ in the first row and then only elements on the diagonal $D(-m/2)$ in the rows from the second to the $(m/2)-1^{th}$. If, on the contrary, we initially choose in the first column the element whose value is $(m/2)-1$:

- it compels the choice of $-((m/2)-1)$ in the last column;
- the choice of elements $(m/2)$ in the columns from 2^{nd} to the $((m/2)-1)^{th}$;
- at this point we can choose only one more element of value $(m/2)-1$ (the same happens in the last $m/2$ columns of D).

6 An algorithm for the general case

In this section we give an algorithm which aims to solve the Problem A for the general case. The algorithm tries to position the elements in the cells of a matrix in such a way that the entire row and column controlled by that element can not be further occupied by other elements. The algorithm uses the technique of backtracking: it tries to locate the item in the first available cell of the first row in the first column, then the second element is placed in the first available cell of the second row in the second column, then the third element so that it does not conflict with the other two, and so on, until all elements are placed in the matrix. For example, if the fifth element was placed in conflict with others, the algorithm relocates the fourth in a new cell and starts with the procedure. Pseudocode for this algorithm is described in the following.

```
Procedure PutElement(col)
row = 1
for each column
  for each row
    if cell is available
      Mark rows and columns as taken
      if columns ¡ MaxElements
    PutElement(col+1)
    Else
      Mark rows and columns as empty
  End for //each row
  Row = row +1  (next row)
End for //each column
Col = col +1
End procedure
```

7 Conclusions

The rules of admissibility/non-admissibility we introduced in sections 4 and 5 allow us to solve, for a fixed m, the problems of existence of a permutation corresponding to vectors $w \ \ ((k+1)_a\ k_b\ \ -k_c\ \ -(k+1)_d)$, in which $a+b>0$ and $c+d>0$, in about a half of the possible cases. For $k<m/4$ general rules appear more difficult to establish, even if many cases are quite easy to solve, particularly when $k=2$ or 3. In certain cases it is more immediate to rely on the geometric representation (i.e. the covering with sub-circuits), while in other cases it is the study of the grid D the major tool. The problem is obviously open, particularly with regard to the classification in terms of computational complexity.

References

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