Advertising and congestion management policies for a museum temporary exhibition

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Summary We propose a model for a museum institution which offers a special exhibition in a definite time period. Attention is given to the museum demand and the quality of the visitors experience, in particular in relation with the occurrence of congestion situations. The laws governing the behaviour of the system through time are defined by three distinct dynamical systems, depending on the position of the visitor attendance rate with respect to two critical levels: the congestion threshold which is a fixed parameter of the system and the extreme congestion threshold which depends on the congestion management policy. Because of the regime switching the optimal control problem is nondifferentiable and Clarke's generalized maximum principle necessary conditions are derived. We discuss the existence of an optimal solution and some special classes of control functions which are economically meaningful.

Keywords: Control, Marketing, Museum, Congestion.

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1. Introduction

The importance of studying the cultural activities from an economic viewpoint has led to the development of the *Cultural Economics* as an autonomous research area [25]. Since the beginning of this approach, which may be defined by the broad analysis of the performing arts carried out by Baumol and Bowen [2], many economists concentrated on both micro and macro aspects of cultural industries (museum institutions, art galleries, organizations which

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operate in the performing arts sector). The economic approach to the cultural sector is not restricted to the financial aspects of cultural industries, such as subsidies and costs. It considers, for example, the role of the cultural heritage in the economic growth of a Nation, as well as the relationship between cultural heritage and tourism; the social aspects of arts enlighted by the use of economic models of human behaviour, as well as the adaptation of the concepts of efficiency and effectiveness to non–profit institutions.

Moreover, as far as museum institutions are concerned, there is a broad economic literature which testifies the strongly increasing interest in them among economists and management scientists ([10], [11], [12], [15], [18], [20], [21]). Nevertheless, only a few mathematical models have been developed in this field and most of them are econometric models.

In [13] we have emphasized the importance of looking into the behaviour of a museum institution from a dynamical systems viewpoint. In [14] we have discussed a variety of modelling elements in order to formulate some specific optimal control problems. The aim of this paper is to propose a special optimal control model in order to analyze the behaviour of a cultural organization in terms of advertising and congestion management policies; the model takes into account some of the basic features of the museum institutions, which can be summarized as follows.

Museums can be seen as special multiobjective firms pursuing some different functions simultaneously, e.g. conservation, exhibition, research and education. As far as the exhibit function is concerned, they generally face the same problems as those of a productive firm which adopts marketing strategies in order to sustain the demand of its products. In fact museum institutions are in competition with a variety of proposals of recreation and entertainment activities for attracting visitors; they may therefore advertise the cultural event in order to enhance the interest for it among the potential audience ([1], [24]). Moreover museums have to build on word–of–mouth communication to sell their services; market research conducted in a museum context confirms that word–of–mouth communication may be more important than advertising (see [18], p.155).

The *exhibition visitor flow* model we propose in this work focuses on the dynamic of museum demand and, in agreement with the above observations, considers two types of communication channel for transmitting information on museum exhibitions within the social system. The first one is media communication, which is directly controlled by the promoter by means of the advertising policy. The second one is word–of–mouth communication, which is related to museum visit conditions and is not affected by the advertising policy.

In the literature concerning optimal control applications to marketing problems (see [9] for a general review) both types of communication processes are considered: for the first type, advertising, relevant references are [19] and [6], whereas for the second type, word–of–mouth communication, we may quote

[7] and [17]. The *exhibition visitor flow model* differs from other dynamic marketing models because it seeks to deal, in a marketing context, with problems of congestion within the museum.

In fact, an important difference between traditional firms and the cultural ones consists in the nature of their products. The services offered by museums have the nature of public goods, with characteristics of non–rivalry and non–excludability in their consumption, to some extent. As a consequence, museum visits, like many public goods (highways, beaches, parks, tourists attractions of all kinds) are subject to crowding and congestion: the presence of other users adversely affects the level of utility obtained by each cultural consumer [22].

Congestion is a subject of considerable importance in outdoor recreation, where more than one agent is attempting to share a type of service that is not supplied in a separable unit for each users. Most contributions to the literature on the management and economic analysis of congestion suggest that physical crowding may cause not only an increase in the time requirement for using a facility but also some subjective effects in terms of deterioration in the quality of the facility and discomfort effects induced by crowding, even though the time costs may not be affected. These subjective congestion effects may be revealed in different forms and they have been dealt with by management scientists, operations researchers and economists from each of their different points of view (see for example [22] and [26]).

We incorporate the subjective effects induced by museum congestion in a general dynamic model that relates the future rate of arrivals to the cumulative number of visitors through a social influence process. In Section 2 we present the *exhibition visitor flow model*, which assumes that the cultural demand is influenced by congestion through the word–of–mouth communication process: past visitors spread both favorable and unfavorable information, according to their museum experience and the occurrence of the unsatisfied visitors is related to the visitor attendance rate being greater than the congestion threshold.

In this way the behaviour of the museum system through time is described by means of a piecewise model involving different dynamical equations in certain regions of the dynamical variables. The optimal control model is non linear and nondifferentiable and the standard Pontryagin maximum principle necessary conditions are not suitable. In Section 3 we present the Clarke's generalized maximum principle necessary conditions for an optimal solution which provide some useful information to define special classes of control functions. We discuss the existence of an optimal solution which is economically meaningful and some special classes of control functions in Section 4 and 5 respectively. Some conclusive remarks are presented in Section 6.

2. The exhibition visitor flow model

2.1. Statement of the exhibition visitor flow problem

The exhibition visitor flow (EVF) problem is the following optimal control problem.

$$\text{maximize} \;\; J \;=\; \int_0^T (\alpha y(t) - v_1(t) - v_2(t))\, dt, \tag{1}$$

subject to

$$\begin{aligned}
\dot{x}(t) &= [\min\{y(t),\, y_C - (1-\eta v_2(t))(y(t)-y_C)\}]^+ && (2.1)\\
\dot{y}(t) &= -\gamma(y(t) - \beta y_C) + a_x x(t) - a_z z(t) + b v_1(t), && (2.2)\\
\dot{z}(t) &= \min\{y(t),\, (2-\eta v_2(t))(y(t)-y_C)^+\}, && (2.3)\\
& x(0)=0, \quad z(0)=0, \quad y(0)=y_0, && (3)\\
& y(T) \geq 0, && (4)\\
& v(t) \in U = [0,\, \overline{v}_1] \times [0,\, \overline{v}_2], && (5)\\
& T \in [0, \overline{T}], && (6)
\end{aligned}$$

where $s^+ \;=\; \max(0,\, s)$.

The variables and parameters have the following meanings:

T, final time, which is the end time of the exhibition $(0 \leq T \leq \overline{T})$;
$\overline{T}$, least upper bound of the feasible final times $(\overline{T} > 0)$;
$y(t)$, visitor attendance rate at time t;
y_C, congestion threshold $(y_C > 0)$;
α, parameter representing the constant admission fee $(\alpha > 0)$;
$x(t)$, cumulative number of satisfied visitors at time t;
$z(t)$, cumulative number of unsatisfied visitors at time t;
$v_1(t)$, advertising expenditure rate at time t $(0 \leq v_1 \leq \overline{v}_1)$;
$\overline{v}_1$, maximum advertising expenditure rate $(\overline{v}_1 > 0)$;
$v_2(t)$, congestion management expenditure rate at time t $(0 \leq v_2 \leq \overline{v}_2)$;
$\overline{v}_2$, maximum congestion management expenditure rate $(\overline{v}_2 > 0)$;
b, parameter representing the influence of the advertising expenditure on the visitor attendance rate $(b > 0)$;
γ, parameter representing the current influence of the visitor attendance rate on itself $(\gamma > 0)$;
β, parameter of current congestion dependence $(0 < \beta \leq 1)$;
η, unsatisfaction enhancement parameter $(0 < \eta \leq \overline{v}_2{}^{-1})$;
a_x, parameter representing the (positive) influence of the actual satisfied visitors on the visitor attendance rate $(a_x > 0)$;
a_z, parameter representing the (negative) influence of the actual unsatisfied visitors on the visitor attendance rate $(a_z > a_x)$;

Museums are assumed to control the advertising expenditure rate and to undertake suitable actions to reduce the effects of congestion on the visitors. On the other hand, the museum's entrance fees will not be treated as a control variable. In fact, usually the public museums cannot vary autonomously their entrance prices.

Equations (2.1) and (2.3) represent the way in which satisfied and unsatisfied visitors appear. All visitors are supposed to be satisfied if the visitor attendance rate is less than the congestion threshold. On the contrary, if the visitor attendance rate is greater than the congestion threshold, then only some visitors are satisfied, whereas the remaining visitors are unsatisfied and the percentages of satisfied or unsatified visitors depend on the congestion management expenditure rate. At each istant of time the following equality holds: $\dot{x}(t) + \dot{z}(t) = y(t)$.

Equation (2.2) represents the growth of museum demand as a function of excess demand, cumulative satisfied visitors, cumulative unsatisfied visitors and advertising. The assumption that $a_z > a_x$ is in agreement with the observation that "negative experiences tend to have a greater impact than positive experiences" ([18], p.155). The museum net benefit rate, which is the integrand function in (1), takes into account that each visitor pays a constant admission fee and brings a constant exhibition cost, but there are congestion management and advertising costs which depend on the museum management decisions.

2.2. Normal, congested and transition regimes

If $v(t)$, $t \in [0, T]$, is a piecewise continuous control function such that $v(t) \in U$, then there exists a unique continuously differentiable state function $(x(t), y(t), z(t))$ which satisfies the motion equations (2) and the initial conditions (3). This follows from the fact that the map $g : \mathbb{R}^5 \to \mathbb{R}^3$ such that $(\dot{x}(t), \dot{y}(t), \dot{z}(t)) = g(x(t), y(t), z(t), v_1(t), v_2(t))$, as defined by (2.1)–(2.3), is a Lipschitz function of (x, y, z), uniformly in (v_1, v_2), since for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ and for all $(v_1, v_2) \in U$, the following inequality holds:

$$\begin{aligned} \|g(x_1, y_1, z_1, v_1, v_2) \; - \; g(x_2, y_2, z_2, v_1, v_2)\| \; \leq \\ \leq \sqrt{2} \max\{3 + \gamma, a_x, a_z\} \|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|. \end{aligned} \tag{7}$$

Hence from a known theorem from calculus (see e.g. [4], p. 577 and [5], p. 274), for all piecewise continuous $(v_1(t), v_2(t))$, the motion equation $(\dot{x}(t), \dot{y}(t), \dot{z}(t)) = g(x(t), y(t), z(t), v_1(t), v_2(t))$ has a unique solution $(x(t), y(t), z(t))$, which is a continuously differentiable function.

We say that the system is in *normal regime* at time t if $y(t) < y_C$, so that the evolution of the satisfed and unsatisfied visitors is determined by the equations:

$$\dot{x}(t) = y(t), \tag{8.1}$$

$$\dot{z}(t) = 0. \tag{8.3}$$

The system is in *simply congested regime* at time t if $y_C < y(t) < y_{EC}(v_2(t))$, where

$$y_{EC}(v_2) \;=\; y_C \frac{2-\eta v_2}{1-\eta v_2}, \tag{9}$$

so that the evolution of the satisfied and unsatisfied visitors is determined by the equations:

$$\dot{x}(t) \;=\; y_C - (1-\eta v_2(t))(y(t)-y_C), \tag{10.1}$$
$$\dot{z}(t) \;=\; (2-\eta v_2(t))(y(t)-y_C). \tag{10.3}$$

The system is in *extremely congested regime* at time t if $y > y_{EC}(v_2(t))$, so that the associated $x(t)$ and $z(t)$ motion equations are:

$$\dot{x}(t) \;=\; 0, \tag{11.1}$$
$$\dot{z}(t) \;=\; y(t), \tag{11.3}$$

i.e. all visitors which find the museum extremely congested remain unsatisfied. If either $y(t) = y_C$ or $y(t) = y_{EC}(v_2(t))$, the system is in *transition regime* at time t.
We observe that $y_{EC}(v_2) \in [2y_C,\, y_{EC}(\overline{v}_2)]$, for all $v_2 \in [0,\, \overline{v}_2]$, so that if $y(t) \in]y_C,\, 2y_C[$, then the system is not extremely congested; if $y(t) \in]y_{EC}(\overline{v}_2),\, \infty[$, then the system is extremely congested.
Moreover, after defining $v_2^{EC}(y)$ as the inverse function of $y_{EC}(\overline{v}_2)$, i.e.

$$v_2^{EC}(y) \;=\; \frac{y-2y_C}{\eta(y-y_C)}, \quad y > y_C, \tag{12}$$

we have that if $y(t) \in]2y_C,\, y_{EC}(\overline{v}_2)[$, then the system is extremely congested as far as $v_2(t) \in [0,\, v_2^{EC}(y)]$; the system is not extremely congested as far as $v_2(t) \in]v_2^{EC}(y),\, \overline{v}_2]$.

Let $(x(t), y(t), z(t), v(t), T)$ be a feasible solution to the museum visitor flow problem, i.e. let $v(t)$, $t \in [0,\, T]$, be a piecewise continuous control function such that $v(t) \in U$ and let $(x(t), y(t), z(t))$ be the state function which is associated to $v(t)$ by the motion equations (2) and the initial conditions (3), so that condition (4) is satisfied. Then we call *epoch* a maximal time interval (w.r.t. $\subseteq$) in which the system is observed staying either in normal regime (a normal epoch), or in simply congested regime (a simply congested epoch), or in extremely congested regime (an extremely congested epoch), or else in transition regime (a transition epoch).

3. Optimality conditions

The standard Pontryagin maximum principle conditions (see [23], pp. 84–86 and p. 143) are not suitable for the EVF problem, because the functions on the r.h.s.'s of equations (2.1) and (2.3) have not continuous partial derivatives w.r.t. y. On the other hand, the Clarke's generalized maximum principle (see [3], pp. 210–212) applies to this problem. The Hamiltonian function is

$$\begin{aligned} H(x,y,z,p_1,p_2,p_3,v_1,v_2,t) \;&=\; p_0(\alpha y - v_1 - v_2) + \\ &+\; p_1\left[\min\{y,\, y_C - (1-\eta v_2)(y-y_C)\}\right]^+ \;+ \\ &+\; p_2(-\gamma(y-\beta y_C) + a_x x - a_z z + b v_1) \;+ \\ &+\; p_3 \min\{y,\, (2-\eta v_2)(y-y_C)^+\}. \end{aligned} \tag{13}$$

If $(x^*, y^*, z^*, v_1^*, v_2^*, T^*)$ is an optimal solution to problem EVF, then there exists a constant $p_0 \in \{0,1\}$ and a continuous and piecewise continuously differentiable function $(p_1(t), p_2(t), p_3(t))$, such that the following conditions hold:

i) for all $t \in [0,T]$,

$$(p_0, p_1(t), p_2(t), p_3(t)) \neq 0; \tag{14}$$

ii) for all $v \in [0,\overline{v}_1] \times [0,\overline{v}_2]$,

$$H(x(t),y(t),z(t),p(t),v^*(t),t) \;\geq\; H(x(t),y(t),z(t),p(t),v,t); \tag{15}$$

iii) if $v(t)$ is continuous at t, then v.e.:

$$\dot{p}_1(t) \;= -p_2(t)a_x; \tag{16}$$

$$\begin{aligned} &\dot{p}_2(t) + \alpha p_0 - \gamma p_2(t) \;= \\ &= \begin{cases} -p_1(t), & y(t) < y_C, \\ -p_3(t) + (p_1(t) - p_3(t))(1-\eta v_2(t)), & \\ & y_C < y(t) < y_{EC}(v_2(t)), \\ -p_3(t), & y(t) > y_{EC}(v_2(t)), \end{cases} \end{aligned} \tag{17.1}$$

and

$$\begin{aligned} &\dot{p}_2(t) + \alpha p_0 - \gamma p_2(t) \;\in \\ &\in \begin{cases} \mathrm{co}\{-p_1(t),\, -p_3(t) + (p_1(t)-p_3(t))(1-\eta v_2(t))\}, \\ \qquad\qquad\qquad\qquad y(t) = y_C, \\ -p_3(t) + \mathrm{co}\{0,\, (p_1(t)-p_3(t))(1-\eta v_2(t))\}, \\ \qquad\qquad\qquad\qquad y(t) = y_{EC}(v_2(t)), \end{cases} \end{aligned} \tag{17.2}$$

where $\mathrm{co}\{r,\, s\} \;=\; [\min(r,\, s),\, \max(r,\, s)]$ is the convex hull of the set $\{r,\, s\}$;

$$\dot{p}_3(t) \;= p_2(t)a_z; \tag{18}$$

iv)

$$p_0 \;\in\; \{0,\, 1\}; \tag{19}$$

v)

$$
\begin{aligned}
p_1(T) &= p_3(T) = 0 \\
p_2(T) &\geq 0 \quad \text{and} \quad p_2(T)y(T) = 0;
\end{aligned}
\tag{20}
$$

vi)

$$
\begin{aligned}
&H(x(T),\, y(T),\, z(T),\, p(T),\, v(T),\, T) \;\geq\; 0, \\
&\text{and} \quad (T - \overline{T}) \cdot H(x(T),\, y(T),\, z(T),\, p(T),\, v(T),\, T) \;=\; 0.
\end{aligned}
\tag{21}
$$

We further observe that the usual Arrow–Kurz type sufficient conditions [8] apply to this problem, so that if there exists a solution to the conditions (14)–(21) with $p_0 = 1$, then it is an optimal solution. ??? In fact, ... ???

We further observe that the usual Arrow-Kurz type sufficient conditions ([8, p.513]) apply to this problem, whenever the control variables either cause no regime switching or cause the system to jump, while occurring the regime transitions at isolated times. In this case, if there exists a solution to the conditions (14)–(21) with $p_0 = 1$, then it is an optimal solution.

In fact, let $(v_1(t), v_2(t), x(t), y(t), z(t))$ be a feasible solution of the EVF problem and let $(p_1(t), p_2(t), p_3(t))$ be such that the optimality conditions (14)-(21) are satisfied, with $p_0 = 1$. Moreover let $(v_1(t), v_2(t))$ causes no regime switching. Then the function

$$H^0 = \max_{v \in U} H$$

is linear in (x, y, z) and then it is concave in the state variables, for every fixed t and $p(t)$. The solution $(v_1(t), v_2(t), x(t), y(t), z(t))$ is then an optimal solution ([8, p.513, Theorem A.3.2]).

....

3.1. Observations on the optimality conditions

¿From the Hamiltonian maximization condition (15) some features of the advertising and management policies may be derived. For the advertising expenditure rate $v_1(t)$ we obtain that

$$v_1(t) = \begin{cases} 0, & p_2(t) < p_0/b, \\ \overline{v}_1, & p_2(t) > p_0/b. \end{cases} \tag{22}$$

On the other hand, for the congestion management expenditure rate $v_2(t)$ the following policies hold

$$v_2(t) = \overline{v}_2, \tag{23.1}$$
$$\text{if either } y_C < y(t) \le 2y_C \text{ and } p_1(t) - p_3(t) > p_0/\eta(y(t) - y_C),$$
$$\text{or } 2y_C < y(t) \le y_{EC}(\overline{v}_2) \text{ and}$$
$$-p_0\overline{v}_2 + (p_1(t) - p_3(t))\,[y_C - (1 - \eta\overline{v}_2)(y(t) - y_C)] > 0;$$

$$v_2(t) = 0, \tag{23.2}$$
$$\text{if either } y(t) \le y_C, \text{ or } y(t) > y_{EC}(\overline{v}_2),$$
$$\text{or } y_C < y(t) \le 2y_C \text{ and } p_1(t) - p_3(t) < p_0/\eta(y(t) - y_C),$$
$$\text{or } 2y_C < y(t) \le y_{EC}(\overline{v}_2) \text{ and}$$
$$-p_0\overline{v}_2 + (p_1(t) - p_3(t))\,[y_C - (1 - \eta\overline{v}_2)(y(t) - y_C)] \le 0.$$

From (23.2) the congestion management expenditure rate is zero during either normal or transition epochs and when $y(t) > y_{EC}(\overline{v}_2)$: in the last case the system is extremely congested no matter which value of $v_2(t)$ were chosen. Moreover, if $y(T) > y_C$, after observing from (20) that

$$p_1(T) - p_3(T) = 0, \quad p_2(T) = 0, \tag{24}$$

we obtain that the congestion management expenditure rate must be zero in a suitable neighborhood of the final time T, so that the museum recipe consists in avoiding to spend to control congestion when the exhibition is ending and the visitor satisfaction cannot influence the future attendance.

Because of analogous continuity reasons an optimal congestion management policy must be zero in suitable neighborhoods of the transition times at which $y(t) = y_C$.

If there exists a solutions to the necessary conditions such that $y(T) > 0$, i.e. such that the exhibition closes with a positive visitor flow, then $p_2(T) = 0$ and $p_0 = 1$, because of (20) and (14). Hence, from (22) we obtain that $v_1(t) = 0$ in a suitable neighborhood of the final time T, as already obtained for $v_2(t)$. Moreover, $H(x(T), y(T), z(T), p(T), v(T), T) = \alpha y(T) > 0$, so that $T = \overline{T}$, i.e. $y(T) > 0$ implies that the exhibition closes at the end of the reserved time interval.

If a piecewise continuous optimal control exists, then it must have a relatively simple structure. In fact, the restriction of such a control function either to a normal, or to a simply congested, or to an extremely congested epoch, must satisfy the Pontryagin maximum principle conditions for special optimal control problems, whose Hamiltonian function are linear w.r.t. the control v. We obtain that a piecewise continuous optimal control must be a bang–bang control, except possibly at transition epochs.

However some restrictive necessary conditions for the control components are known, even in a transition epoch:

a) if $v_1(t) \in]0, \overline{v}_1[$ for all $t \in]\sigma, \tau[$, then

$$p_2(t) = p_0/b, \quad \text{for all } t \in]\sigma, \tau[; \tag{25}$$

b) if $v_2(t) \in]0, \overline{v}_2[$ for all $t \in]\sigma, \tau[$, then

$$p_1(t) - p_3(t) = \frac{p_0}{\eta[y(t) - y_C]} \text{ and } y_C < y(t) < 2y_C, \quad \text{for all } t \in]\sigma, \tau[. \tag{26}$$

Moreover, a transition epoch is most likely a singleton set, as different restrictive conditions should hold if it were a nondegenerate interval.

4. Existence of an optimal solution

As far as the existence of an optimal solution is concerned, we have a positive result in the context of measurable controls and we can state that there exists a measurable control function which maximizes the objective functional of the EVF problem. In fact, the integrand function in the objective functional is a continuously differentiable function in (x, y, z), for all v_1, v_2 and t. Therefore, the EVF problem is equivalent to the following EVF^* problem:

$$\begin{aligned} \text{maximize} \;\; J &= w(T), && (27)\\ \text{subject to} \;\; \dot{w}(t) &= (\alpha y(t) - v_1(t) - v_2(t)), && (28)\\ w(0) &= 0, && (29)\\ \text{and to} \;\; & (2.1),\ (2.2),\ (2.3),\ (3),\ (4),\ (5),\ (6). \end{aligned}$$

The functions which determine the motion equations do not depend explicitly on the time variable t and, for all $x, y, z \in R$, the set

$$\left\{ \begin{pmatrix} [\min\{y,\, y_C - (1 - \eta v_2)(y - y_C)\}]^+ \\ -\gamma(y - \beta y_C) + a_x x - a_z z + b v_1 \\ \min\{y,\, (2 - \eta v_2)(y - y_C)^+\} \\ (\alpha y - v_1 - v_2) \end{pmatrix} \in R^4 \mid v \in U \right\} \tag{30}$$

is a segment of R^4, hence it is a convex set. Moreover, for all measurable v and for all $t \in [0, T]$, where $T \leq \overline{T}$ is the final time of the solution associated with the control v, we have that

$$
\begin{aligned}
0 \ &\leq \ x(t) \ \leq \ y_C \overline{T}, && (31.1)\\
0 \ &\leq \ y(t) \ \leq \ \left[\gamma\beta y_C + a_x y_C \overline{T} + b\overline{v}_1\right] \overline{T}, && (31.2)\\
0 \ &\leq \ z(t) \ \leq \ \left[\gamma\beta y_C + a_x y_C \overline{T} + b\overline{v}_1\right] \overline{T}^2, && (31.3)\\
|w(t)| \ &\leq \ \overline{T} \max\left\{\overline{v}_1 + \overline{v}_2,\, \alpha\left[\gamma\beta y_C + a_x y_C \overline{T} + b\overline{v}_1\right] \overline{T}\right\}. && (31.4)
\end{aligned}
$$

This two facts allow us to use a known theorem on the compactness of the set of attainability (Theorem 2, p. 242, in [16]), which holds also in the case that the functions which define the motion equations are just Lipschitz functions and state the existence of an optimal measurable control.

5. A special class of control functions

In the following we present a few examples of control functions and discuss them with respect to their practical significance and their analytic features.

In the dynamical system (2.1)-(2.3), there is a stable equilibrium for the state function $y(t)$ whenever we consider a constant control $v(t) \equiv (\hat{v}_1, \hat{v}_2)$, or equivalently the control $v(t) = w(t, \hat{v}_1, \hat{v}_2)$ such that

$$
\begin{aligned}
w_1(t) \ &= \ \hat{v}_1, \quad t \in [0, T];\\
w_2(t) \ &= \ \begin{cases} 0, & \text{if } y(t) \notin \,]y_C,\, y_{EC}(\hat{v}_2)[,\\ \hat{v}_2, & \text{if } y(t) \in \,]y_C,\, y_{EC}(\hat{v}_2)[. \end{cases}
\end{aligned}
\tag{32}
$$

The equilibrium value is

$$
\hat{y}(\hat{v}_2) \ = \ \frac{(2 - \eta\hat{v}_2)(a_x + a_z)}{a_x(1 - \eta\hat{v}_2) + a_z(2 - \eta\hat{v}_2)} y_C \ \in \,]y_C,\, y_{EC}(\hat{v}_2)[, \tag{33}
$$

which depends on $\hat{v}_2$ but not on $\hat{v}_1$. A congestion management policy can modify both the eventual level of congestion and the unsatisfied visitors increment rate. On the contrary, every advertising policy is ineffective in the long run and only the word-of-mouth communication matters.

Now, let us define the control $v(t)$ as a variant of $w(t, \overline{v}_1, \overline{v}_2)$ and such that

$$
\begin{aligned}
v_1(t) \ &= \ \begin{cases} \overline{v}_1, & \text{if } t \in [0, \sigma[,\\ 0, & \text{if } t \notin [0, \sigma[, \end{cases}\\
v_2(t) \ &= \ \begin{cases} 0, & \text{if } y(t) \notin \,]y_1,\, y_2[\text{ or } t \notin [0, \tau[,\\ \overline{v}_2, & \text{if } y(t) \in \,]y_1,\, y_2[\text{ and } t \in [0, \tau[, \end{cases}
\end{aligned}
\tag{34}
$$

for some fixed $\sigma, \tau \in [0, \overline{T}[$ and $y_1,\, y_2$, such that $y_1 \in]y_C,\, 2y_C[$ and $y_2 \in]y_1,\, y_{EC}(\overline{v}_2)[$.

The control functions defined by (34) share some general features with the solutions to the optimality necessary conditions. In fact, for all controls of this kind

a) extreme values only are observed, $v(t) \in \{0, \overline{v}_1\} \times \{0, \overline{v}_2\}$,
b) the congestion management expenditure rate is zero in suitable neighborhoods of the transition times at which $y(t) = y_C$,
c) if the final time is large enough, $T > \max\{\sigma, \tau\}$, then both the advertising expenditure rate and the congestion management expenditure rate are zero in suitable neighborhoods of the final time T.

A control defined by (34) depends on time and on the state of the system (but not simply on the regime of the system). The associated state function $(x(t), y(t), z(t))$ is continuously differentiable at all times except those t at which $y(t) \in \{y_1, y_2\}$: the latter are isolated points. The transition epochs are singleton sets. The control may not be equivalent to any piecewise continuous control, as it is possible that $y(t) \in \{y_1, y_2\}$ at infinitely many points in a bounded interval, thus implying that $v_2(t)$ would have infinitely many switches in a bounded interval. Such controls are not optimal, but practically good solutions may be found among them. They represent a museum management which advertises the cultural event during an initial segment of the time interval $[0, T]$ and opposes the congestion during certain subintervals of the simply congested epochs.

5.1. "Congestion overlooking" policy

If $\sigma \geq 0$ and $\tau = 0$, then the control function represents a congestion overlooking policy, with maximum advertising effort in $[0, \sigma[$, if $\sigma > 0$; whereas we have the zero policy $v(t) \equiv 0$, if $\sigma = \tau = 0$. The control drives the system towards the equilibrium $\hat{y}(0)$ for the state function $y(t)$. The system will possibly have regime transitions, at isolated times, as follows:

* from the normal regime to the simply congested regime ($N \to SC$), or from the simply congested regime to the normal regime ($SC \to N$), at times t such that $y(t) = y_C$;
* from the simply congested regime to the extremely congested regime ($SC \to EC$), or from the extremely congested regime to the simply congested regime ($EC \to SC$), at times t such that $y(t) = 2y_C$.

5.2. "Partial relief of simple congestion" policy

If $\sigma \geq 0$ and $\tau > 0$, then the state function $(x(t), y(t), z(t))$ will possibly have regime transitions, in the final segment $[\tau, T]$, at isolated times with the same characteristics as for the zero policy. Moreover, the state function $y(t)$ will tend to $\hat{y}(0)$ in $[\tau, T]$. On the other hand, to study its behaviour in the initial segment $[0, \tau]$, we can distinguish two particular cases.

Let $y_2 \in]y_1, 2y_C[$ and $y_1 \in]y_C, 2y_C[$, then the associated state function $(x(t), y(t), z(t))$ will possibly have regime transitions, at isolated times in the initial segment $[0, \tau]$, as follows:

* $N \to SC$ or $SC \to N$ at times t such that $y(t) = y_C$;
* $SC \to EC$ or $EC \to SC$ at times t such that $y(t) = 2y_C$.

The control $v_2(t)$ will have switches at times t such that $y(t) \in \{y_1, y_2\}$.
There is an equilibrium for the state function $y(t)$ at $\hat{y}(0)$, if $\hat{y}(0) \in]y_C, y_1[\cup]y_2, 2y_C[$, and at $\hat{y}(\overline{v}_2)$, if $\hat{y}(\overline{v}_2) \in]y_1, y_2[$.

prsc/2

Let $y_2 \in [2y_C, y_{EC}(\overline{v}_2)]$ and $y_1 \in]y_C, 2y_C[$, then the associated state function $(x(t), y(t), z(t))$ will possibly have regime transitions, at isolated times in the initial segment $[0, \tau]$, as follows:

* $N \to SC$ or $SC \to N$ at times t such that $y(t) = y_C$;
* $SC \to EC$ or $EC \to SC$ at times t such that $y(t) = y_2$ and which are switch times for the control.

The control $v_2(t)$ will have further switches at times t such that $y(t) = y_1$.
There is an equilibrium for the state function $y(t)$ at $\hat{y}(0)$, if $\hat{y}(0) \in]y_C, y_1[$, and at $\hat{y}(\overline{v}_2)$, if $\hat{y}(\overline{v}_2) \in]y_1, y_2]$.

In both particular cases, we observe that there may exist 0, 1, or 2 equilibria for the state function $y(t)$. In fact, there does not exist any equilibrium if $y_1 < \hat{y}(0) < y_2 < \hat{y}(\overline{v}_2)$, whereas there exist two equilibria if $\hat{y}(0) < y_1 < \hat{y}(\overline{v}_2) < y_2$.

Therefore, if we want that a *prsc* policy allow the system to have a definite equilibrium for $y(t)$, during the interval $[0, \tau]$, then we remain with two families of *prsc* policies:

* *expensive prsc* policy: $y_1 \in]y_C, \hat{y}(0)[$ and $y_2 \in]\hat{y}(\overline{v}_2), y_{EC}(\overline{v}_2)[$, which has the unique equilibrium $\hat{y}(\overline{v}_2)$ in $[0, \tau]$;
* *cheap prsc* policy: $y_1 \in]\hat{y}(0), \hat{y}(\overline{v}_2)[$ and $y_2 \in]y_1, \hat{y}(\overline{v}_2)[$, which has the unique equilibrium $\hat{y}(0)$.

6. Conclusions

The model of the exhibition visitor flow we have introduced takes into account some of the basic features of the cultural institutions. We have formulated the cultural marketing problem of determining optimal advertising policies for a museum institution with the objective of maximizing a profit functional. It is a nonlinear and nondifferentiable optimal control problem with three state and two control variables. The behaviour of the museum system through time has been described by means of three dynamical systems. The first one is associated to a low attendance rate (normal regime), when no actual congestion effects are observed. The second one is associated to a high attendance rate (simply congested regime), when the visitor satisfaction is affected negatively, though not cancelled, by the actual congestion. The third one is associated to a very high attendance rate (extremely congested regime), when the visitor satisfaction is cancelled by the actual congestion.

We have obtained the Clarke's generalized maximum principle necessary conditions for an optimal solution, which here do not allow to determine explicitly any solution. We have also proved the existence of an optimal measurable control. Some information from the necessary conditions has been used to define special classes of control functions which are economically meaningful. We have analysed the characteristics of such special controls and the dependence of the system behaviour on them, also by means of some numerical simulations. It appears that some heuristic method may be designed in order to find good control functions in suitable classes of controls as those we have studied.

Further attention could be given to the modelling of the visitor congestion sensitivity, which is essential for the evolution of the system state. We think that the motion equations (2.1) and (2.2) may be modified in order to account for different degrees of visitor congestion sensitivity, without changing the qualitative behaviour of the system or the significance of the control functions analyzed.

References

[1] Ames PJ (1989) Marketing in museums: means or master of the mission? Curator 32: 5–15.

[2] Baumol WJ, Bowen WG (1966) Performing Arts: the Economic Dilemma. MIT Press, Cambridge.

[3] Clarke FH (1983) Optimization and Nonsmooth Analysis. Wiley, New York.

[4] Comincioli V (1990) Analisi Numerica – metodi, modelli, applicazioni. McGraw Hill, Milano.

[5] De Marco G (1993) Analisi Due/2. Decibel, Padova.

[6] Dockner E, Jorgensen S (1988) Optimal advertising policies for diffusion models of new product innovation in monopolistic situations. Management Science 34: 119–130.

[7] Dodson JA, Muller E (1978) Models of new product diffusion through advertising and word–of–mouth. Management Science 24: 1568–1578.

[8] Feichtinger G, Hartl RF (1986) Optimale Controlle Ökonomischer Prozesse. Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften. Berlin, Walter de Gruyter.

[9] Feichtinger G, Hartl RF, Sethi SP (1994) Dynamic optimal control models in advertising: recent developments. Management Science 40: 195–226.

[10] Feldstein M (1991) The Economics of Art Museums. The University of Chicago Press, Chicago.

[11] Frey BS (1994) Cultural economics and museum behaviour. Scottish Journal of Political Economy 41: 325–335.

[12] Fronville LC (1985) Marketing for museums: for–profit techniques in the non–profit world. Curator 28: 169–182.

[13] Funari S, Viscolani B (1999) Allocazione dinamica delle risorse in un museo: il problema della gestione della congestione. Economia della Cultura VIII: 169–183.

[14] Funari S, Viscolani B (2000) Dynamic modelling of a cultural event. Rendiconti per gli Studi Economici Quantitativi, to appear.

[15] Jackson R (1988) A museum cost function. Journal of Cultural Economics 12: 41–50.

[16] Lee EB, Markus L (1968) Foundations of Optimal Control Theory. Wiley, New York.

[17] Mahajan V, Muller E, Kerin RA (1984) Introduction strategy for new products with positive and negative word–of–mouth. Management Science 30: 1389–1404.

[18] McLean F (1997) Marketing the Museum. Routledge, London.

[19] Nerlove M, Arrow KJ (1962) Optimal advertising policy under dynamic conditions. Economica 29: 229–232.

[20] O'Hare M (1974) The public's use of art–visitor behavior in an art museum. Curator 17: 309–320.

[21] Peacock A, Godfrey C (1974) The economics of Museums and Galleries. Lloyds Bank Review 111: 17–28.

[22] Rothenberg J (1970) The economics of congestion and pollution: an integrated view. American Economic Review 60: 114–121.

[23] Seierstad A, Sydsaeter K (1987) Optimal control theory with economic applications. North–Holland, Amsterdam.

[24] Stone DL (1988) The use of Advertising to attract the museum public. Curator 31: 123–130.

[25] Throsby D (1994) The Production and Consumption of the Arts: a View of Cultural Economics. Journal of Economic Literature 33: 1–29.

Additional reference

[26] Van Ackere A. (1996) The management of congestion. European Journal of Operational Research 89: 223–225.