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Variations on the measure representation approach

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Abstract

The measure approach represents a preference relation over functions by the measure of their epigraphs (or hypographs). This paper proves a measure representation theorem for a class of increasing functions and shows how its proof can be modified to yield another measure representation theorem for functions of bounded variation. © 1998 Elsevier Science B.V.

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1. Introduction and theme

The measure approach to the representation of preferences over lotteries was proposed by Segal (1989; first version 1984) as a generalization of the rank-dependent model initiated by Quiggin (1982) and Schmeidler (1989; first version 1982).

According to (a version of) the rank-dependent model, a monetary lottery X on the positive reals with cumulative distribution F can be evaluated by the functional

$$\mathrm{RDEU}(X) = \int u(x)\, \mathrm{d}(g \circ F)(x), \tag{1}$$

where $g:[0,1] \to [0,1]$ is strictly increasing and onto. This form reduces to the

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special case of expected utility when the probability transformation function g is the identity, but it still separates attitudes to outcomes, modeled through u, and attitudes to probabilities. modeled through g.

Segal (1989) pointed out that the rank-dependent model may be interpreted as if the underlying preference relation $\succeq$ over lotteries is represented by a (product) measure of the epigraphs of the cumulative distribution functions. In fact, assume without loss of generality that $u(0)=0$ and note that the two increasing functions u and g are defined respectively on the outcomes axis and on the probability axis, so that we can compute the measure of the rectangle $[x_1, x_2) \times [p_1, p_2)$ by the product $[u(x_2)-u(x_1)] \cdot [g(p_2)-g(p_1)]$.

Consider a lottery X with a finite number of possible outcomes $0 \leq x_1 \leq \cdots \leq x_n$. Let $F(x_0)=0$ and (1) reduces to

$$\mathrm{RDEU}(X) = \sum_{i=1}^{n} [(g \circ F)(x_i) - (g \circ F)(x_{i-1})] u(x_i),$$

which is precisely the sum of the measures of the rectangles $[0, x_i) \times [F(x_{i-1}), F(x_i))$. The union of these rectangles constitutes the epigraph of F truncated from above at the level $p=1$.

Following this interpretation, a natural extension of the rank-dependent model is to represent the preference relation $\succeq$ over lotteries by a general (not necessarily product) measure of the truncated epigraphs. This idea was pursued in Segal (1989), Green and Jullien (1988) and Chew and Epstein (1989) but the statements of their representation theorems were incorrect, as shown by the counterexamples in Wakker (1993). Later on, Wakker (1993a) gave a thorough discussion of the difficulties related to the measure representation approach and correct proofs were offered in Segal (1993), Chew et al. (1993) and Chateauneuf (1996: first version 1990). Recently, Chew and Wakker (1996) generalized the representation from lotteries to acts.

The simple idea of a measure representation may be applied for preferences over many relevant classes of functions. See Chateauneuf (1985) and Lehrer (1991) for a very general approach, closely related to the theory of qualitative probability. Among the possible examples, we mention: (i) cumulative distribution functions or, alternatively, probability densities; (ii) cumulated cash flows or, alternatively, gain/loss processes; (iii) income distribution functions on the real line or alternatively, concentration curves; and (iv) investment or, alternatively, consumption profiles for a single good over time.

Most of these examples, however, refer to functions that are not necessarily increasing and therefore require versions of the measure representation theorem different from the ones available in the literature. So far, instead, the measure representation approach has been mainly concerned with preferences over lotteries represented by their cumulative distribution functions, i.e. with bounded, increasing and right-continuous functions.

The main purpose of this paper is to prove a similar result for a more general class of functions as one of the many possible 'variations' on the theme of the measure representation approach.

The paper is organized as follows. The first variation is a theorem for the measure representation of preferences over increasing functions. The second variation presents some technical comments on this result and prepares for the third and last variation, which shows how to modify the proof given for increasing functions and get a measure representation theorem for preferences over functions of bounded variation.

2. Variation 1: Increasing functions

We begin with some notation. Let $\mathscr{F} = \{F, G, \dots\}$ be the set of all increasing and right-continuous functions defined on $[0,1)$, taking values in $[0,1)$, and different from the constant zero function. Given a function F in $\mathscr{F}$, we denote by $x, x', \dots$ the elements of its domain and by $y, y', \dots$ the elements of its image.

Suppose that there is a preference relation (i.e., a complete weak order) $\succeq$ on $\mathscr{F}$ satisfying the following three properties:

Continuity. The preference relation $\succeq$ is continuous in the topology of the weak convergence. That is, for any pair of functions F and G, suppose that the sequence of functions $\{F_n\}$ weakly converges to F (i.e., $F_n(x)$ converges to $F(x)$ at each continuity point x of F): then $F_n \succeq G$ for all n implies $F \succeq G$ and $G \succeq F_n$ for all n implies $G \succeq F$.

Strict monotonicity. The preference relation $\succeq$ is strictly monotone with respect to the pointwise order. That is, for any pair of functions F and G, suppose that $F(x) \geq G(x)$ for all x and there exists at least some x_1 such that $F(x_1) > G(x_1)$; then $F \succ G$, where $\succ$ is defined as usual.

Independence (w.r.t. the graph). The preference relation $\succeq$ is independent of common pieces of the function graphs. That is, for any segment S in $[0,1)$ and any quadruple F, F', G and G', suppose that $F(x) = G(x)$ and $F'(x) = G'(x)$ on S and that $F(x) = F'(x)$ and $G(x) = G'(x)$ on $[0,1) \setminus S$; then $F \succeq G$ if and only if $F' \succeq G'$.

Recall that a finitely additive and extended positive real-valued function μ on an algebra $\mathscr{A}$ of subsets of a set X is called a *measure* on X. A measure μ on X is said to be *continuous from above* if, for all sequences of sets A_n in $\mathscr{A}$ such that $A_n \downarrow \emptyset$, $\mu(A_n) \downarrow 0$. Whenever $\mathscr{A}$ is a σ-algebra, a measure μ on X is countably additive if and only if it is continuous from above. A (countably additive) measure μ on X is said to be *σ-finite* if there exists a sequence of sets A_n in $\mathscr{A}$ such that $\mu(A_n) < \infty$ for all n and $\bigcup_n A_n = X$.

We will prove a measure representation theorem for $\succeq$. Intuitively, its content is the following. Denote by R the rectangle $[0,1) \times [0,1)$. The graph of each function F in $\mathscr{F}$ is contained in R and uniquely defines its (truncated) hypograph $\hat{F} = \{(x, y) \in R : y < F(x)\}$. We can put the set of functions $\mathscr{F}$ in a one-to-one relation with the set $\hat{\mathscr{F}}$ of their truncated hypographs, so that each function F in $\mathscr{F}$ corresponds to its hypograph $\hat{F}$ in $\hat{\mathscr{F}}$. Continuity, strict monotonicity, and independence (w.r.t. the graph) will be shown to imply the existence of a measure μ on R such that $F \succeq G$ if and only if $\hat{F}$ is bigger than $\hat{G}$.

If we knew μ to be finite, this statement would be formalized as

$$F \succeq G \text{ if and only if } \mu(\hat{F}) \geq \mu(\hat{G}). \tag{2}$$

However, it turns out that we can only prove that μ is σ-finite. Therefore. we need a stronger formulation that implies (2) when μ is finite. Thus, we will show that continuity, strict monotonicity, and independence (with respect to the graph) imply the existence of a (countably additive and σ-finite) measure μ on R such that $F \succeq G$ if and only if $\mu(\hat{F} \setminus \hat{G}) \geq \mu(\hat{G} \setminus \hat{F})$.

Alternatively, Segal (1993) suggests the following (equivalent) formulation. For any $z \in [0,1)$, let $\mathscr{F}^z = \{F \in \mathscr{F} : F(x) \geq z \text{ for } x > 1 - z\}$ be a subset of functions in $\mathscr{F}$ and $\hat{\mathscr{F}}^z$ the set of their truncated hypographs. Let $Z = (1 - z,1) \times [0, z)$. Denote by μ^z the measure on R defined by $\mu^z(A) = \mu(A \setminus Z)$. For all z, continuity, strict monotonicity, and independence (w.r.t. the graph) imply the existence of a (countably additive and finite) measure μ^z on R such that μ^z represents $\succeq$ on $\mathscr{F}^z$; that is, for every F and G in $\mathscr{F}^z$, $F \succeq G$ if and only if $\mu^z(\hat{F}) \geq \mu^z(\hat{G})$.

Some more definitions are necessary before we state the theorem. A *curve* C in R is the image of a continuous bijective[1] function $f : [0,1) \to R$; note that a curve does not necessarily belong to $\mathscr{F}$. A curve C in R is increasing if $(x_1, y_1) \in C$ implies that the intersection between C and the northwest region $\{(x_2, y_2) \in R : x_2 < x_1 \text{ and } y_2 > y_1\}$ is empty. A *rectangle* is any set $[x_1, x_2] \times [y_1, y_2)$ in R such that $x_1 < x_2$ and $y_1 < y_2$; note that a rectangle is the product of two intervals closed on the left and open on the right and that by definition it has nonempty interior.

The following characterization theorem is very similar to Theorem 1 in Segal (1993), but our setting has some advantages that will be discussed in the second variation. The proof is in three parts. For the first two (easier) parts, we follow Segal (1993). For the harder proof of the third part, we exploit an argument similar to the one put forth in Green and Jullien (1988) and Chew and Epstein (1989) that has also been used in the literature on rank-dependent models.

[1] The standard definition of curve does not require bijectivity. We impose it because, by Netto's theorem, this rules out the possibility of space-filling curves. See for instance Theorem 1.3 in Sagan (1994).

Theorem 1. The following three statements are equivalent:

1. *A preference relation $\succeq$ on $\mathscr{F}$ satisfies continuity, strict monotonicity and independence w.r.t. the graph.*
2. *There exists a (countably additive and σ-finite) measure μ on R such that, for all z,*
   (a) *the preference relation induced by $\succeq$ on $\mathscr{F}^z$ is represented by μ^z;*
   (b) *μ^z is countably additive and finite;*
   (c) *μ assigns strictly positive measure to any rectangle in R;*
   (d) *μ assigns zero measure to any increasing curve in R.*
3. *There exists a measure μ on R as in Proposition 2 satisfying (c) and (d) and such that $F \succeq G$ if and only if $\mu(\hat{F} \setminus \hat{G}) \geq \mu(\hat{G} \setminus \hat{F})$.*

Proof. We prove that Proposition 2 implies Proposition 3, which in turn implies Proposition 1, which in turn implies Proposition 2.

Proposition 2 implies Proposition 3. Let F and G be in $\mathscr{F}$. By definition of $\mathscr{F}$, there exists $\varepsilon > 0$ such that $\lim_{x \uparrow 1} F(x) > \varepsilon$ and $\lim_{x \uparrow 1} G(x) > \varepsilon$. Therefore, by right-continuity, there exists $\varepsilon' > 0$ such that $\min\{F(x), G(x)\} > 0$ for $x \geq 1 - \varepsilon'$. For $z = \min\{\varepsilon, \varepsilon'\}$, then, F and G are in $\mathscr{F}^z$ and thus

$$F \succeq G \text{ if and only if } \mu^z(\hat{F}) \geq \mu^z(\hat{G}) \text{ if and only if } \mu\left(\hat{F} \setminus Z\right) \geq \mu\left(\hat{G} \setminus Z\right).$$

Since $\hat{F} \setminus Z = (\hat{F} \setminus \hat{G}) \cup [(\hat{F} \cap \hat{G}) \setminus Z]$, this implies by additivity of μ that $F \succeq G$ if and only if $\mu(\hat{F} \setminus \hat{G}) \geq (\hat{G} \setminus \hat{F})$. □

Proposition 3 implies Proposition 1. Strict monotonicity follows from (c) and independence w.r.t. the graph from the measure representation. It remains to prove that $\succeq$ is continuous in the topology of the weak convergence. It suffices to show that, for all sequences F_n weakly converging to F, $\mu(\hat{F}_n)$ converges to $\mu(\hat{F})$.

Let $S_n = (\hat{F}_n \cup \hat{F}) \setminus (\hat{F}_n \cap \hat{F})$ be the symmetric difference between the hypographs of F and F_n. Let $T_n = \bigcup_{k=n}^{\infty} S_k$. Note that $\{T_n\}$ is a decreasing sequence of sets: let $T = \lim_n T_n = \bigcap_n T_n$. Since $|\mu(\hat{F}_n) - \mu(\hat{F})| \leq \mu(S_n) \leq \mu(T_n)$, we only need to show that $\lim_{n \to \infty} \mu(T_n) = 0$. This will follow if we prove that $\mu(T) = 0$. In fact, since μ is countably additive and $\{T_n\}$ is a decreasing sequence[2] of sets, continuity from above of μ implies that $\lim_n \mu(T_n) = \mu(T)$; see for instance Theorem 10.2.ii in Billingsley (1986).

Thus, it remains to show that $\mu(T) = 0$. Let F^* be the northwest boundary of $\hat{F}$: that is, $F^* = \{(x,y) \in \mathrm{cl}(\hat{F}) : x' < x \text{ and } y' > y \text{ implies } (x',y') \notin \hat{F}\}$. Since F^* is an increasing curve, $\mu(F^*) = 0$. We show that $T \subset F^*$ and therefore $\mu(T) = 0$.

Suppose by contradiction that $T \not\subset F^*$. Then there exists some point (x', y') in T which is not contained in F^*. There are two possible cases: either $y' > F(x')$ or

[2] By the σ-finiteness of μ, it also holds that $\mu(T_1) < \infty$.

$y' < \lim_{x \uparrow x'} F(x)$. Moreover, since (x',y') is in $T = \bigcap_n T_n = \bigcap_n \bigcup_{k=n}^{\infty} S_k = \limsup S_k$, there exists a subsequence (which we index by n as the original sequence) $\{F_n\}$ such that $(x',y') \in S_n$.

If $y' > F(x')$, then $(x',y') \notin \hat{F}$. Thus, by definition of S_n, $(x',y') \in \hat{F}_n$ and $\lim_{x \downarrow x'} F_n(x) > y'$ for all n. Since functions in $\mathscr{F}$ are increasing and right-continuous, there exists $\varepsilon > 0$ such that $F(x) < [F(x') + y']/2 < y' < F_n(x)$, for all $x \in [x', x' + \varepsilon)$. As the interval $[x', x' + \varepsilon)$ must contain a continuity point of F, F_n cannot be weakly converging to F.

Similarly, if $y' < \lim_{x \uparrow x'} F(x)$, then $(x',y') \in \hat{F}$; thus $(x',y') \notin \hat{F}_n$; and $\lim_{x \uparrow x'} F_n(x) \le y'$ for all n. Then there exists $\varepsilon > 0$ such that $F_n(x) \le y' < [\lim_{x \uparrow x'} F(x) + y']/2 < F(x)$, for all $x \in (x' - \varepsilon, x']$. Again, the interval $(x' - \varepsilon, x']$ must contain a continuity point of F and F_n cannot be weakly converging to F. $\square$

Proposition 1 implies Proposition 2. This part of the proof is organized in four steps that are conceptually simple, but somewhat cumbersome due to the number of definitions involved. An overview of the proof is the following. First, we prove the existence of a real-valued functional V defined on a subset $\mathscr{F}_r$ of simple functions in $\mathscr{F}$ such that $V(F) \ge V(G)$ if and only if $F \succeq G$ for all F and G in $\mathscr{F}_r$. Second we show that for any z this functional induces a representing measure μ on the algebra $\mathscr{A}_r^z$ generated by the hypographs of the functions in $\mathscr{F}_r^z = \mathscr{F}_r \cap \mathscr{F}^z$ and that μ is σ-finite and satisfies (c) of Theorem 1; that is, for any z a measure representation holds for $\mathscr{F}_r^z$. Third, we check that μ extends to the σ-algebra generated by the hypographs of all functions in $\mathscr{F}$ and that this extension is σ-additive. Fourth, we prove that for any z this extension provides a measure representation for all functions in $\mathscr{F}^z$ and satisfies (d) of Theorem 1.

Step 1. Existence of a representing functional on $\mathscr{F}_r$.

A function F in $\mathscr{F}$ is said to be *simple* if its image is a finite subset of [0,1). Note that the image of a simple function in $\mathscr{F}$ cannot be the singleton {0}. Using the indicator function $\mathbf{1}\{\cdot\}$, a simple function can be written as a finite sum $F(x) = \sum_{i=1}^{n} F(x_i) \cdot \mathbf{1}\{[x_i, x_{i+1})\}$ with $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1 \equiv x_{n+1}$ with at least one inequality holding strictly. The ranked n-tuple of points $x_1, x_2, \ldots, x_n$ used in this representation is called a *nth dimensional basis* for F and is denoted by $\mathbf{x}$. A simple function admits an infinite number of bases; however, there always exists a smallest one that we call its *minimal basis*.

We say that a simple function F in $\mathscr{F}$ is *simpler* if its minimal basis contains only rational numbers of the type $k/2^n$ for some $n = 2,3,\ldots$ and $k = 1,\ldots,2^n - 1$. Let $\mathscr{F}_r$ be the set of all simpler functions in $\mathscr{F}$ and $\mathscr{F}_r^n$ be the subset of all simpler functions such that their minimal basis contains only rational numbers of the type $k/2^n$ for $k = 1,\ldots,2^n - 1$ and $F(2^{n-1}/2^n) > 0$. Note that $\mathscr{F}_r^n \subset \mathscr{F}_r^m$ for all $n < m$ and that $\mathscr{F}_r = \lim_n \mathscr{F}_r^n = \bigcup_n \mathscr{F}_r^n$.

Consider a class $\mathscr{F}_r^{\,n}$ and an arbitrary element F in $\mathscr{F}_r^{\,n}$. For each $k = 1,\ldots,2^n - 1$, let $y_k = F(k/2^n)$ and put each function F in $\mathscr{F}_r^{\,n}$ in a one-to-one relation with the $(2^n - 1)$-tuple $\mathbf{y}$ which lists its image points $0 \le y_1 \le y_2 \le \cdots \le y_{2^n-1} < 1$ (possibly with repetitions and with at least one weak inequality holding strictly). Denote by I_r^n the set of these rank-ordered $(2^n - 1)$-tuples of reals representing the functions in $\mathscr{F}_r^{\,n}$. Note that the two $(2^n - 1)$-tuples consisting respectively only of 0's and 1's do not belong to I_r^n. These would represent the *extreme alternatives* which are respectively inferior and superior to any element in I_r^n; thus I_r^n contains no extreme alternatives.

The restriction of the preference relation $\succeq$ to $\mathscr{F}_r^{\,n}$ defines another preference relation $\succeq_n$ on I_r^n that inherits the (equivalent of the) properties of $\succeq$. First, $\succeq_n$ is a complete weak order. Second, it is continuous with respect to the product topology on I_r^n. Third, $\succeq_n$ is strictly increasing in the following sense: write $\mathbf{y}_{-k}\alpha$ for $\mathbf{y}$ in I_r^n with the kth coordinate y_k replaced by α in [0,1); then $\alpha > \beta$ implies $\mathbf{y}_{-k}\alpha \succ \mathbf{y}_{-k}\beta$ for all α and β in [0,1). Fourth, $\succeq_n$ satisfies *coordinate independence*: $\mathbf{y}_{-k}\alpha \succeq_n \mathbf{y}'_{-k}\alpha$ if and only if $\mathbf{y}_{-k}\beta \succeq_n \mathbf{y}'_{-k}\beta$.

By Theorems 3.2 and 3.3 in Wakker (1993a), then, there exist $(2^n - 1)$ strictly increasing and continuous real-valued functions $V_k^n : [0,1) \to \mathbb{R}$ (for $k = 1,2,\ldots,2^n - 1$) such that $V^n(\mathbf{y}) = \sum_k V_k^n(y_k)$ represents $\succeq_n$ on I_r^n. Moreover, the functions $\{V_n^k\}$ are *jointly cardinal*; that is, they can be replaced by $\{W_k^n\}$ if and only if there exist real $\alpha_1,\ldots,\alpha_{2^n-1}$ and a positive β such that $W_k^n = \alpha_k + \beta V_k^n$ for all k.

Through the one-to-one relation between $\mathscr{F}_r^{\,n}$ and I_r^n, the functional $V^n : I_r^n \to \mathbb{R}$ also represents the preference relation $\succeq$ on $\mathscr{F}_r^{\,n}$. It remains to prove that there exists a functional representing $\succeq$ on $\mathscr{F}_r$. Since $\mathscr{F}_r = \bigcup_n \mathscr{F}_r^{\,n}$, this will be obtained by 'pasting' together the representations for each $\mathscr{F}_r^{\,n}$

Each set I_r^n has (more than) one representing functional V_n for $\succeq_n$. However, since $\mathscr{F}_r^{\,n} \subset \mathscr{F}_r^{\,m}$ for $n < m$ and $\succeq_n$ agrees with $\succeq_m$ on $\mathscr{F}_r^{\,n}$, we can use the degrees of freedom provided by joint cardinality to normalize the functions $\{V_k^n\}$ for all $n = 2,3,\ldots$ and $k = 1,\ldots,2^n - 1$ so that: (i) $V_k^n(0) = 0$, and hence $V^m(\mathbf{0}) = V^n(\mathbf{0})$ for $m > n$; and (ii) if $m > n$,

$$\sum_{k=1}^{2^n-1} V_k^n(y) = \sum_{k=1}^{2^m-1} V_k^m(y) \quad \text{for all } y \in [0,1)$$

so that whenever $\mathbf{y}^n$ is an $(2^n - 1)$-tuple with $y_k = y$ for all k, $V^m(\mathbf{y}) = V^n(\mathbf{y})$.

Let φ^n be the real-valued functional on $\{0,1/2^n,2/2^n,\ldots,(2^n-1)/2^n\} \times [0,1)$ defined by

$$\varphi^n(0,y) = 0 \text{ and } \varphi^n\left(\frac{k}{2n}, y\right) = \sum_{s=1}^{k} V_s^n(y)$$

for all $n = 2,3,\ldots$ and $k = 1,2,\ldots,2^n - 1$. Note that

$$\sum_{k=1}^{2^n-1}\left[\varphi^n\left(\frac{k}{2^n}, y_k\right) - \varphi^n\left(\frac{k-1}{2^n}, y_k\right)\right] = \sum_{k=1}^{2^n-1} V_k^n(y_k) = V^n(y) \tag{3}$$

represents $\succeq_n$ on $\mathscr{F}_r^{\,n}$. Moreover, by the above normalization, φ^n is positive with $\varphi^n(x,0)=0$ for all n and all x in $\{0,1/2^n,2/2^n,\dots,(2^n-1)/2^n\}$.

Denote by Q the set of rational numbers of the type $k/2^n$ for some $n=2,3,\dots$ and $k=1,\dots\ 2^n-1$. After the above normalization,

$$\varphi^n\left(\frac{k}{2^n},y\right)=\varphi^m\left(\frac{k}{2^n},y\right),$$

for all $m>n$ and thus, applying the definition of φ^n for all n, we obtain a positive real-valued functional φ defined on $Q\times[0,1)$ such that: (i) the form given in (3) represents $\succeq$ on $\mathscr{F}_r$; (ii) $\varphi(x,\ 0)=0$ for all x in Q; and (iii) $\varphi(0,y)=0$ for all y in $[0,1)$. $\square$

Step 2. A measure representation exists for $\mathscr{F}_r^{\,z}$.

Given z, we wish to show that there exists a representing measure μ defined on the algebra $\mathscr{A}_r^z$ generated by the hypographs of the simpler functions in $\mathscr{F}_r^{\,z}$. Observe that the hypograph of a simple function is the union of a finite number of rectangles. For a simpler function, these rectangles are of the form $[x,x')\times[y,y')$ with $x=(k_1/2^{n_1})$ and $x'=(k_2/2^{n_2})$. Without loss of generality, we take $n_1=n_2=n$. These rectangles will be called *evenly rational*. Denote by $\mathscr{A}_r$ the algebra generated by all evenly rational rectangles and by $\mathscr{A}_r^z$ the algebra generated by the differences between all evenly rational rectangles and the southeast corner $Z=(z,1)\times[0,z)$. Note that $\mathscr{A}_r=\lim_{z\uparrow 1}\mathscr{A}_r^z=\bigcup_z\mathscr{A}_r^z$.

The basic tool to construct a measure μ on $\mathscr{A}_r^z$ is the positive functional φ on $Q\times[0,1)$. It satisfies the following property, known as total strict positivity of order 2 or strict supermodularity:

$$\varphi(x',y')-\varphi(x,y')-\varphi(x',y)+\varphi(x,y)>0, \tag{4}$$

for all $x<x'$ in Q and $y<y'$ in $[0,1)$. In fact, for $x=(k_1/2^n)<(k_2/2^n)=x'$,

$$\begin{aligned}
&[\varphi(x',y')-\varphi(x,y')]-[\varphi(x',y)-\varphi(x,y)]\\
&=\left[\sum_{s=1}^{k_2}V_s^n(y_s')-\sum_{s=1}^{k_1}V_s^n(y_s')\right]-\left[\sum_{s=1}^{k_2}V_s^n(y_s)-\sum_{s=1}^{k_1}V_s^n(y_s)\right]\\
&=\sum_{s=k_1+1}^{k_2}V_s^n(y_s')-\sum_{s=k_1+1}^{k_2}V_s^n(y_s)\\
&=\sum_{s=k_1+1}^{k_2}\left[V_s^n(y_s')-V_s^n(y_s)\right]>0,
\end{aligned}$$

where the inequality follows from strict increasingness of V^n.

For every (evenly rational) rectangle $[0,x)\times[0,y)$ in $\mathscr{A}_r^z$, define its measure by

$$\nu^z([x_1,x_2)\times[y_1,y_2)) = \varphi(x_2,y_2) - \varphi(x_1,y_2) - \varphi(x_2,y_1) + \varphi(x_1,y_1). \qquad (5)$$

This gives a finite measure ν_z on $\mathscr{A}_r^z$ characterized by the two-dimensional distribution function $\varphi(x,y) = \nu^z([0,x)\times[0,y))$. Moreover, for any $z<w$, $\mathscr{A}_r^z \subset \mathscr{A}^w$ and $\nu^z(A) = \nu^w(A)$ if A is in $\mathscr{A}_r^z$. Therefore, applying the definition of ν_z for all z, we obtain a measure μ on $\mathscr{A}_r$.

By (4), μ is strictly positive on any rectangle in R. Moreover, since for any increasing (countable) sequence $\{z_n\}$ converging to 1 we have $\mu([0,z_n]\times[z_n,1)) < \infty$, the measure μ is also σ-finite.

It remains to prove that μ^z represents $\succeq$ on $\mathscr{F}_r^z$ for any z. This follows easily because the hypograph $\hat{F}$ of each function F in $\mathscr{F}_r^z$ is an element of $\mathscr{A}_r^z$ and the representing functional given in (3) computes $\mu_z(\hat{F})$. $\square$

Step 3. Extension of μ to a countably additive measure on a σ-algebra.

The extension of μ from $\mathscr{A}_r$ to the σ-algebra generated by $\mathscr{A}_r$ is a standard procedure: see for instance Billingsley (1986), Theorem 12.5. It is based on the Caratheodory extension theorem, that we state in a version especially convenient for our setting; see for instance Aliprantis and Border (1994), Theorem 8.40.

Lemma 2. Let μ be a measure continuous from above on an algebra $\mathscr{A}$ of subsets of R. Then μ extends to a countably additive measure on the σ-algebra generated by $\mathscr{A}$. Moreover, if μ is σ-finite on $\mathscr{A}$, this extension is unique.

Denote by $\mathscr{A}$ the algebra generated by all (not necessarily evenly rational) rectangles. The σ-algebra generated by $\mathscr{A}_r$ and $\mathscr{A}$ is the same, namely the Borel σ-algebra $\mathscr{B}$ for R. By the first part of Lemma 2, proving that there exists a countably additive extension of μ to $\mathscr{B}$ requires only to show that μ is continuous from above or, more simply, that φ is continuous (separately) in each argument.

Continuity of $\varphi(x,y)$ with respect to $y\in[0,1)$ follows from continuity of $V_k^n(y)$ for all n and k. Continuity with respect to $x\in Q$ is obvious for $y=0$ because $\varphi(x,y)=0$ for all x and follows by continuity of $\succeq$ if $y>0$. In fact, suppose by contradiction that x_n converges to x but $\varphi(x_n,y)$ does not converge to $\varphi(x,y)$. Without loss of generality, assume that there exists $z>x$ such that $\max_n x_n < z$. Define $F_n = y\cdot\mathbf{1}\{[x_n,1)\}$ and $F = y\cdot\mathbf{1}\{[x,1)\}$. Then F_n weakly converges to F, but $\mu^z(\hat{F}_n) = \varphi(z,y) - \varphi(x_n,y)$ does not converge to $\mu^z(\hat{F}) = \varphi(z,y) - \varphi(x,y)$. Since $\{F_n\}$ and F are in $\mathscr{F}_r^z$ and μ^2 represents $\succeq$ on $\mathscr{F}_r^z$, this would contradict continuity of $\succeq$.

The following implication of the continuity of φ will be useful. A subset $[x,x')\times\{y\}$ in R is called a *horizontal segment*; similarly, a subset $\{x\}\times[y,y')$

in R is called a *vertical segment*. The (from now, extended) measure μ assigns zero measure to any horizontal or vertical segment. For instance, the measure of the horizontal segment $[x,x') \times \{y\}$ is

$$\mu([x,x') \times y) = \lim_{y' \downarrow y} \mu([x,x') \times [y,y'))$$

$$= \lim_{y' \downarrow y} [\varphi(x',y') - \varphi(x',y)] - [\varphi(x,y') - \varphi(x,y)] = 0$$

and a similar derivation holds for vertical segments. □

Step 4. For all z, a measure representation exists for $\mathscr{F}^z$.

Given z, we wish to show that the measure μ^z represents $\succeq$ on $\mathscr{F}^z$. First, we check that $\hat{\mathscr{F}}^z$ is contained in $\mathscr{B}$. Note that the hypographs of all continuous function (even those not in $\mathscr{F}^z$) are contained in $\mathscr{B}$. Since each function F in $\mathscr{F}^z$ is increasing, it has at most a countable number of discontinuities. Therefore its hypograph $\hat{F}$ is the union of at most a countable number of hypographs of continuous functions; hence, $\hat{F}$ belongs to $\mathscr{B}$.

Take now F in $\mathscr{F}^z \setminus \mathscr{F}_r^z$. Observe that F is the limit (in the topology of the weak convergence) of some sequence $\{F_n\}$ in $\mathscr{F}_r^z$. Therefore, by continuity of $\succeq$, it should be assigned the measure $\lim_n \mu_z(\hat{F}_n)$ for the measure representation to hold. Hence, we need to show that

$$\mu^z(\hat{F}) = \lim_n \mu^z(\hat{F}_n), \tag{6}$$

for any sequence F_n converging to F.

Given any such sequence, note that $F_n(x)$ may fail to converge at $F(x)$ only in a (possibly empty) countable set $D = \{x_1, x_2, \dots\}$ of discontinuity points of F. If D is empty, then $\hat{F} = \lim_n \hat{F}_n$ and (6) holds by continuity of μ. If D is not empty, for any x_k in D let v_k the vertical segment $\{x_k\} \times [0,1)$. Then

$$|\mu^z(\hat{F}) - \lim_n \mu^z(\hat{F}_n)| \leq \mu^z(\cup_k v_k) = \sum_k \mu^z(v_k) = 0$$

and again (6) follows.

There remains only to prove that μ satisfies (d) of Theorem 1. First, note that any increasing curve C is in $\mathscr{B}$ because it can always be identified with the northwest boundary of an appropriate function F in $\mathscr{F}$. Next, suppose by contradiction that there exists an increasing curve C such that $\mu(C) > 0$. Given F in $\mathscr{F}$ corresponding to C, choose a rectangle $Z = (z,1) \times [0,z)$ such that $Z \cap C = \varnothing$ and let $\{F_n\}$ be a sequence in $\mathscr{F}$ converging from above to F (in the topology of the weak convergence) and such that $C \subset \hat{F}_n$ for all n. Since $C \cap \hat{F}$ contains at most vertical segments that have zero measure, we can assume without loss of generality that $C \cap \hat{F} = \varnothing$. Then we have that $\mu^z(\hat{F}_n \setminus \hat{F}) \geq \mu^z(C) > 0 = \mu^z(\hat{F} \setminus \hat{F}_n)$. But this implies that $\mu^z(\hat{F}_n) = \mu^z(\hat{F}_n \setminus \hat{F}) + \mu^z(\hat{F} \cap \hat{F}^n)$ cannot converge to $\mu^z(\hat{F}) = \mu^z(\hat{F} \setminus \hat{F}_n) + \mu^z(\hat{F} \cap \hat{F}_n)$, which is impossible. □

It is worth pausing on the main difference between our proof of Theorem 1 and the proofs of similar results given in Green and Jullien (1988) and Chew and Epstein (1989). Our construction of the representing functional on $\mathscr{F}^r$ in Step 1 partitions the x-axis instead of the y-axis. This simple trick, analogous to the intuition that makes Lebesgue integration more general than Riemann integration, will be crucial during our third variation.

We close this variation making two observations. First, the representing measure μ is unique up to a ratio scale; that is, up to a positive scale factor. Second, as shown by the counterexamples in Wakker (1993), μ may not be absolutely continuous with respect to the Lebesgue measure λ on R and, conversely, λ may not be absolutely continuous with respect to μ.

3. Variation 2: Technical remarks

Theorem 1 may be generalized in many directions. This section considers and comments on some of these possibilities. Recall that the functions in $\mathscr{F}$ satisfy the following four properties: they are (i) increasing; (ii) right-continuous; (iii) defined on $[0,1)$ and taking values in $[0,1)$; and (iv) different from the constant zero function. Leaving for the next section a relaxation of (i), let us examine here how (ii)–(iv) might be weakened.

We begin with right-continuity. Intuitively, its purpose is to exclude the possibility of two functions $F \succ G$ in $\mathscr{F}$ with their hypographs $\hat{F}$ and $\hat{G}$ coinciding everywhere except for an (at most) countable set of vertical segments. In this case, the measure μ could not represent $F \succ G$ because of property (d). Obviously, we might substitute left-continuity for right-continuity.

More generally, we could start with a set $\mathscr{F}'$ of functions containing $\mathscr{F}$ and define an equivalence relation $\sim$ on $\mathscr{F}'$ such that $F \sim G$ if and only if $\mathrm{cl}(\hat{\mathscr{F}}) = \mathrm{cl}(\hat{G})$. Then each right-continuous function in $\mathscr{F}$ is the representative element of a (distinct) equivalence class. Assuming that $\succeq$ on $\mathscr{F}'$ is consistent with $\sim$, Theorem 1 would yield a representing measure for $\succeq$ on $\mathscr{F}'/\sim$. All in all, however, right-continuity (or left-continuity) appear the most reasonable assumptions.

Properties (iii) and (iv), instead, have the purpose of excluding front $\mathscr{F}$ the two extreme alternatives, namely the constant zero and one functions. This is necessary in order to apply Theorem 3.3 from Wakker (1993a) during Step 1 of the proof that Proposition 1 implies Proposition 2. The two properties could be removed if the second-order Archimedean axiom for $\succeq$ described in Section 3.3 of Wakker (1993a) would be added. However, as we find this axiom difficult to interpret, we prefer to avoid it and impose some restrictions on the domain and the range of the functions in $\mathscr{F}$.

These restrictions must be such that $\mathscr{F}$ contains no extreme alternatives. Among the many possibilities, our choice to set the domain and the range both equal to [0,1) is only a convenient normalization. In fact, we could substitute [3] any pair of clopen (and bounded from below) intervals $[a_1,b_1) \times [a_2,b_2)$ in $\mathbb{R}^2$ and, in particular, $a_i = 0$ and $b_i = +\infty$ for $i = 1,2$. This gives the class of positive (possibly unbounded) real-valued functions defined on $\mathbb{R}^+$. As these are often used to model cumulated cash flows, for instance, Theorem 1 may be applied as it is to yield a measure representation for (positive and increasing) cumulated cash flows.

All the measure representation theorems so far presented in the literature are inspired by the rank-dependent model and refer to (truncated) epigraphs. Theorem 1, instead, is based on (truncated) hypographs. An obvious comment is that this is a consequence of the direction in which strict monotonicity holds. More interestingly, however, there is a simple way to turn Theorem 1 into a representation result based on epigraphs.

For any increasing function F in $\mathscr{F}$, define its (generalized) inverse φ by $\varphi(y) = \inf\{x : F(x) > y\}$. Let Φ be the set of all generalized inverses of $\mathscr{F}$ and let $\succeq'$ be the preference relation on Φ induced by $\succeq$. Note that the strict monotonicity of $\succeq$ for $\mathscr{F}$ is increasing with respect to the pointwise order on $\mathscr{F}$, while the strict monotonicity of $\succeq'$ for Φ is decreasing. Continuity and independence with respect to the graph, instead, carry over naturally without modifications. Therefore, the measure representing $\succeq$ on $\mathscr{F}$ via hypographs also represents $\succeq'$ on Φ via epigraphs and the two approaches are equivalent.

Our last comment concerns the possibility to get another (more natural) measure representation for functions taking both positive and negative values. We present one example that should suffice to clarify the issue. Let $\mathscr{F}$ be the set of all increasing and right-continuous functions defined on [0,1), taking values in $[-1,1)$ and different from the constant function $F(x) = -1$. Suppose that there exists a preference relation $\succeq$ on $\mathscr{F}$ satisfying continuity, strict monotonicity and independence w.r.t. the graph. Then there exists a measure μ on $R = [0,1) \times [-1,1)$ that represents $\succeq$.

Fix some z [0,1) and define μ^z on $R \setminus Z$ as above. Decompose R into the union of the two disjoint sets $R^+ = [0,1) \times [0,1)$ and $R^- = [0,1) \times [-1,0)$. Define a finite signed measure ν^z on R by

$$\nu^z(A) = \mu^z(A \cap R^+) - \mu^z(R^- \setminus A),$$

for any μ-measurable set A. Since μ^z is finite, we have

$$\begin{aligned}\mu^z(\hat{F}) &= \mu^z(\hat{F} \cap R^+) + \mu^z(\hat{F} \cap R^-)\\ &= \mu^z(\hat{F} \cap R^+) + \mu^z(R^-) - \mu^z\left(R^- \setminus \hat{F}^-\right) = \nu^z(\hat{F}) + \mu^z(R^-).\end{aligned}$$

Since $\mu^z(R^-)$ is constant, we can represent $\succeq$ on $R \setminus Z$ by the signed measure ν^z: that is, by the difference between the measure of the hypograph of the positive part $F^+ = \max\{F,0\}$ (truncated between 0 and 1) and the measure of the epigraph of the negative part $F^- = \min\{F,0\}$ (truncated between -1 and 0).

[3] It suffices to map continuously [0,1) into $[a_i, b_i)$.

4. Variation 3: Functions of bounded variation

This section shows how the proof of Theorem 1 can be generalized to obtain a measure representation for functions of bounded variation. The crucial ingredient is having constructed the representing functional on $\mathscr{F}^r$ in Step 1 by partitioning the x-axis instead of the y-axis. We will point out during the proof the key point where this is used.

Unless explicitly mentioned, we keep the same notation as above. Let $\mathscr{F}$ be the set of all right-continuous functions of bounded variation F defined on [0,1), taking values in [0,1), and such that $\lim_{x \uparrow 1} F(x) > 0$. Assume that $\succeq$ is a preference relation on $\mathscr{F}$.

The theorem that we present in this section is the analog of Theorem 1, except for (d) which requires the following definition. A curve C in R is *not backward bending* if its intersection with any vertical segment in R is either a singleton or a vertical segment. Note that any increasing curve is also not backward bending.

Theorem 3. The following three statements are equivalent:

1. *A preference relation $\succeq$ on $\mathscr{F}$ satisfies continuity, strict monotonicity and independence w.r.t. the graph.*
2. *There exists a (countably additive and σ-finite) measure μ on R such that, for all z,*
   - *(a) the preference relation induced by $\succeq$ on $\mathscr{F}^z$ is represented by μ^z;*
   - *(b) μ^z is countably additive and finite;*
   - *(c) μ assigns strictly positive measure to any rectangle in R;*
   - *(d) μ assigns zero measure to any curve in R which is not backward bending.*
3. *There exists a measure μ on R as in Proposition 2 satisfying (c) and (d) and such that $F \succeq G$ if and only if $\mu(\hat{F} \setminus \hat{G}) \geq \mu(\hat{G} \setminus \hat{F})$.*

Proof. Again, we prove that Proposition 2 implies Proposition 3. which in turn implies Proposition 1, which in turn implies Proposition 2. The proof that Proposition 2 implies Proposition 3 carries over identically from Theorem 1 and thus we omit it.

Proposition 3 implies Proposition 1. Assume that $\succeq$ is continuous in the topology of pointwise convergence and proceed as in the proof of Theorem 1 until when it remains to show that $\mu(T) = 0$. Let F^* be the upper boundary of $\hat{F}$; that is, F^* is the set of all pairs (x,y) in cl$(\hat{F})$ such that: (i) either $x' < x$ and $y' > y$ implies $(x',y') \notin F\}$, or (ii) $x' > x$ and $y' > y$ implies $(x',y') \notin \hat{F}\}$. Then F^* is not a backward bending curve and $\mu(F^*) = 0$. We show that $T \setminus F^*$ is a subset of a (possibly empty) countable union of vertical segments, so that $\mu(T \setminus F^*) = 0$. Then $\mu(T) = \mu(T \setminus F^*) + \mu(T \cap F^*) = 0$.

Suppose that $T \setminus F^*$ is not empty (otherwise the claim follows immediately). Let (x',y') be a point in T which is not contained in F^*. There are two possible cases: (i) $y' > \max\{F(x'), \lim_{x \uparrow x'} F(x)\}$; or (ii) $y' < \min\{F(x'), \lim_{x \uparrow x'} F(x)\}$. Denote by $\{F_n\}$ the subsequence for which $(x', y') \in S_n$.

If $y' > \max\{F(x'), \lim_{x \uparrow x'} F(x)\}$, then $(x', y') \notin \hat{F}$. Thus, $(x', y') \in \hat{F}_n$ and $F_n(x') > y'$ for all n. Then $F(x') < y' < F_n(x')$ for all n implies that x' must be a discontinuity point of F.

Similarly, if $y' < \min\{F(x'), \lim_{x \uparrow x'} F(x)\}$, then $(x', y') \in \hat{F}$. Thus, $(x', y') \notin \hat{F}_n$ and $F_n(x') \le y'$ for all n. Then $F_n(x') \le y' < F(x')$ for all n implies that x' must be a discontinuity point of F.

Recall that each function of bounded variation can be written as the difference of an increasing function and a decreasing function and thus it has at most a countable number of discontinuities. Denote by D the set of discontinuity points of F. We have just shown that any point (x', y') contained in $T \setminus F^*$ must be such that $x' \in D$. Therefore, if we let v_k denote the vertical segment corresponding to each x_k in D, it follows that $T \setminus F^* \subset \bigcup_k v_k$. $\square$.

Proposition 1 implies Proposition 2. Proceed as in the proof of Theorem 1 up until the definition of I_r^n. Note that the $(2^n - 1)$-tuple $\mathbf{y}$ which is to be put in a one-to-one relation with each function F in $\mathscr{F}_r^{\,n}$ does not necessarily satisfy $0 \le y_1 \le y_2 \le \cdots y_{2^n} - 1 < 1$ because F may not be increasing. Instead, we can only say that $y_i \in [0,1)$ for all i. Therefore, let I_r^n the set of all (not necessarily rank-ordered) $(2^n - 1)$-tuples of reals representing the functions in $\mathscr{F}_r^{\,n}$. Again. the restriction of $\succeq$ to $\mathscr{F}_r^{\,n}$ defines a preference relation $\succeq_n$ on I_r^n that inherits the properties of $\succeq$.

Thus, instead of Wakker's, we can call upon the additive utility theorems of Debreu (1960) and Gorman (1968) – here is the key point – to deduce the existence of $(2^n - 1)$ strictly increasing and continuous real-valued functions V_k^n such that $V^n(\mathbf{y}) = \sum_n V_k^n(y_k)$ represent $\succeq_n$ on I_r^n. The rest of the proof goes on unchanged until Step 4, where we need to check that $\hat{\mathscr{F}}^w$ is contained in $\mathscr{B}$. This follows because each function in $\mathscr{F}$ is of bounded variation and thus has at most a countable number of discontinuities. Now, we can resume the argument in the proof of Theorem 1 up until to when we need to show that (d) holds. Note that any curve C which is not backward bending is in $\mathscr{B}$ because it can be identified with the upper boundary of some appropriate function F in $\mathscr{F}$. The rest of the argument goes through unchanged. $\square$

We close this paper by noting a corollary of Theorem 3 that relies on some of the technical remarks in our second variation. For $a < 0$, let $\mathscr{F}$ be the set of all real-valued and right-continuous functions F of bounded variation on $[a, +\infty)$ such that $F(+\infty) = \lim_{x \to +\infty} F(x) > a$. For F in $\mathscr{F}$, denote by $\hat{F}^+$ the (truncated) hypograph of F^+ and by $\hat{F}^-$ the (truncated) epigraph of F^-. Finally, given F and G in $\mathscr{F}$, let $z < \min\{F(+\infty), G(+\infty)\}$ and say that F and G are in $\mathscr{F}^z$.

Corollary 4. The following two statements are equivalent:

1. *A preference relation $\succeq$ on $\mathscr{F}$ satisfies continuity, strict monotonicity and independence w.r.t. the graph.*

2. *There exists a (countably additive and σ-finite) measure* μ *on R that satisfies* (*c*) *and* (*d*) *of Theorem 3 and, for all functions F, G in* $\mathscr{F}^z$, *defines another (countably additive and finite) measure* μ^z *such that*

$$F \succeq G \text{ if and only if } \mu^z(\hat{F}^+) - \mu^z(\hat{F}^-) \geq \mu^z(\hat{G}^+) - \mu^z(\hat{G}^-).$$

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