# Discussion on "Of Quantiles and Expectiles: Consistent Scoring Functions, Choquet Representations, and Forecast Ranking" by W. Ehm, T. Gneiting, A. Jordan, and A. Krüger, written by Roberto Casarin† and Francesco Ravazzolo‡ (†University Ca' Foscari of Venice, ‡Free University of Bozen-Bolzano)

The authors are to be congratulated on their excellent intuition, which has culminated in the development of a quite general approach to consistent scoring of competing forecasting models. Their approach based on extremal scoring functions is in the spirit of the existing literature and in particular of the seminal papers of Gneiting and Ranjan (2011) and Gneiting and Ranjan (2013). We believe that the proposed consistent scoring functions can be applied to achieve new density calibration and density combination schemes.

Consider the forecast distributions $F_1$ and $F_2$ from two predictive models and $F$ the distribution of $Y$. Following the notation used by the authors, one could consider the following map

$$(\theta, \xi) \mapsto D(\theta, \xi) = \mathbb{E}_{\mathbb{Q},\xi}(S_{\alpha,\theta}(X, Y)) \tag{1}$$

where $X$ is a point forecast, $\alpha$ a quantile level, $\theta \in \Theta \subset \mathbb{R}$ a threshold parameter and $\xi \in \Xi \subset \mathbb{R}^k$ a combination/calibration parameter vector. If the parameter $\xi$ is indexing a family of distributions $H_{\xi,F_1}(X) = (g_\xi \circ F_1)(X)$, with $x \mapsto g_\xi(x) \in (0, 1)$ a family of non-decreasing functions with $g(0) = 0$ and $g(1) = 1$, then we have a calibration scheme. If $\xi$ is indexing the family of distributions $H_{\xi,F_1,F_2}(X) = \nu F_1(X) + (1 - \nu)F_2(X)$ then we obtain a combination scheme. Optimal calibration and optimal combination can be defined as

$$\theta \mapsto \inf_{\xi \in \Xi} D(\theta, \xi) \tag{2}$$

As a first example we consider the perfect (solid line) and sign-reversed (dashed-dotted line) forecasters given in Tab. 2 of the paper. We assume a beta calibration scheme and let $g_\xi(x) = B(x; \mu\phi, (1 - \mu)\phi)$ be the incomplete beta function with parameters $\xi = (\mu\phi, (1 - \mu)\phi)$, and $q_{\xi(\theta),\alpha} = z_{\xi(\theta),\alpha} - \mu$ the quantile of the calibrated sign-reversed model where $z_{\xi(\theta),\alpha} = \Phi^{-1}(g^{-1}_{\xi(\theta)}(\alpha))$ and $\xi(\theta)$ is a solution of Eq. 2. Then the Murphy diagram of the calibrated model is $D(\theta) = -\alpha\Phi(q_{\xi(\theta),\alpha} - \theta) + \alpha(1 - \Phi(\theta/\sqrt{2})) + \int_{-\infty}^{z_{\xi(\theta),\alpha}} \Phi(\theta - z)\varphi(z)dz$ and is given in Fig. 1 As a second example we consider two biased forecasters $F_1$ and $F_2$ with distribution $N(-1, 1)$ and $N(2, 2)$, respectively, and use the extremal scoring rule to find the optimal combination. The Murphy diagram of the optimal predictive pooling model is $D(\theta) = -\alpha\Phi(\theta/\sqrt(2)) + \alpha(1 - \mathbb{I}(q_{\xi(\theta),\alpha} > \theta) + \mathbb{I}(q_{\xi(\theta),\alpha} > \theta)\Phi(\theta/\sqrt(2))$ (see Fig. 2), where $q_{\xi,\alpha}$ is the $\alpha$-quantile of $\nu\Phi(q + 1) + (1 - \nu)\Phi((q - 2)/\sqrt{2})$.

In their paper, the authors sketch a number of possible extensions. We recommend as further exciting and stimulating research line the use of consistent scoring functions for model combination and/or model calibration with the aim to improve upon Mitchell and Hall (2005), Hall and Mitchell (2007), Geweke and Amisano (2010), Geweke and Amisano (2011), Billio et al. (2013), Fawcett et al. (2015). We are therefore very pleased to be able to propose the vote of thanks to the authors for their work.

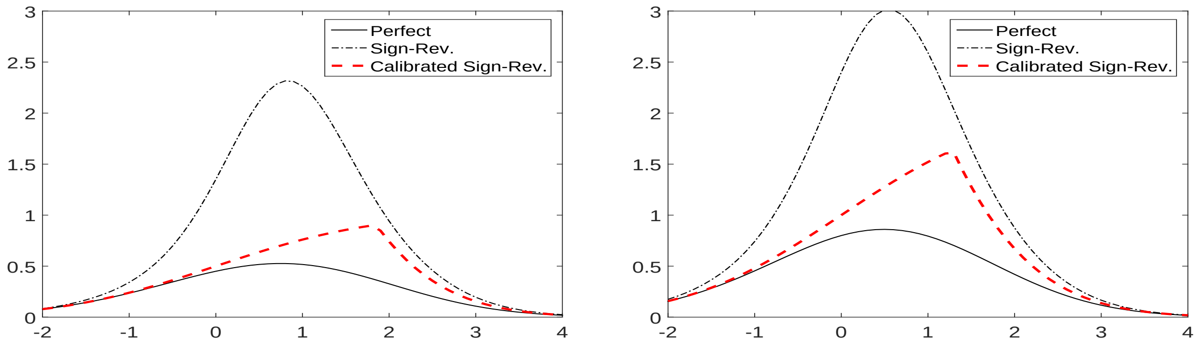


Figure 1: Murphy diagrams for $\alpha = 0.9$ (left) and $\alpha = 0.8$ (right), for the perfect (solid line) and sign-reversed (dashed-dotted line) forecasters given in Tab. 2 of the paper, and for the calibrated forecaster obtained by applying a beta calibration function to the sign-reversed forecaster (red dashed line).

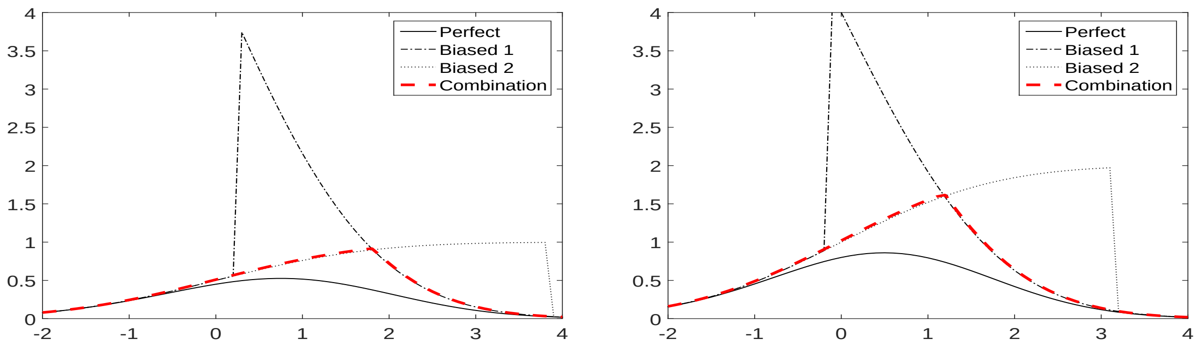


Figure 2: Murphy diagrams for $\alpha = 0.9$ (left) and $\alpha = 0.8$ (right), for the perfect (solid line), biased (dashed-dotted and dotted lines), and combined (red dashed line) forecasters.

# References


Billio, M., Casarin, R., Ravazzolo, F., and Van Dijk, H. (2013). Time-varying combinations of predictive densities using nonlinear filtering. *Journal of Econometrics*, 177:213–232.

Fawcett, N., Kapetanios, G., Mitchell, J., and Price, S. (2015). Generalised density forecast combinations. *Journal of Eonometrics*, 188:150–165.

Geweke, J. and Amisano, G. (2010). Comparing and evaluating Bayesian predictive distributions of asset returns. *International Journal of Forecasting*, 26:216–230.

Geweke, J. and Amisano, G. (2011). Optimal prediction pools. *Journal of Econometrics*, 164:130–141.

Gneiting, T. and Ranjan, R. (2011). Comparing density forecasts using threshold and quantile weighted proper scoring rules. *Journal of Business and Economic Statistics*, 29:411–422.

Gneiting, T. and Ranjan, R. (2013). Combining predictive distributions. *Electronic Journal of Statistics*, 7:1747–1782.

Hall, S. G. and Mitchell, J. (2007). Combining density forecasts. *International Journal of Forecasting*, 23:1–13.

Mitchell, J. and Hall, S. G. (2005). Evaluating, comparing and combining density forecasts using the KLIC with an application to the Bank of England and NIESER "fan" charts of inflation. *Oxford Bulletin of Economics and Statistics*, 67:995–1033.