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Silvia Faggian
and
Giuseppe Freni

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A Ricardian Model of Forestry

Silvia Faggian
University of Venice at Ca' Foscari

Giuseppe Freni
University of Naples Parthenope

Abstract
This paper provides a continuous-time "Ricardian" model of forestry, where, in response to an increase in timber demand, forest cultivation is progressively intensified on the most fertile lands and/or extended to less fertile qualities of lands. It is shown that, at a given level of the rate of interest, a set of "break-through timber prices" gives the *order of fertility* (i.e., the order in which the different qualities of land are taken into cultivation) and that, for each land, prices of standing trees are positive above a "threshold timber price". Since, for each land, the break-through price is higher than the threshold price, Ricardo is shown to be right: a higher demand for timber could simply raise those components of the landlord compensation which are not rent.

Keywords: Vintage Capital, Ricardian extensive rent theory, Harvesting problems, Forest Management

JEL Codes: C61, C62, E22, D90, Q23.

Address for correspondence:
Silvia Faggian
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
Phone: (++39) 041 2346913
e-mail: faggian@unive.it

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wp.dse@unive.it

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

1 Introduction

Ricardo's *Principles* contain just a single reference to forestry, as a whole, although in a well-known and remarkable passage in Chapter II, "On Rent". There, a rigorous notion of *rent* is introduced, and Adam Smith is criticized for using the term in an inconsistent way:

> Adam Smith [...] tells us that the demand for timber, and its consequent high price, in the more southern countries of Europe, caused a rent to be paid for forests in Norway, which could before afford no rent. Is it not however evident, that the person who paid, what he thus calls rent, paid it in consideration of the valuable commodity which was then standing on the land, and that he actually repaid himself with a profit, by the sale of the timber? If, indeed, after the timber was removed, any compensation were paid to the landlord for the use of the land, for the purpose of growing timber or any other produce, with a view to future demand, such compensation might justly be called rent, because it would be paid for the productive powers of the land; but in the case stated by Adam Smith, the compensation was paid for the liberty of removing and selling the timber, and not for the liberty of growing it.
>
> Ricardo (1951, p. 68)

Then Ricardo seems to contemplate two different possible effects of a higher demand for timber: the extension of forestry to new lands on a permanent base, which could lead to the rise of the rents paid on these lands, and a temporary rise of timber production due to the extraction of timber from standing trees. In two recent insightful papers, Kurz and Salvadori (Kurz & Salvadori, 2009, 2011) emphasize the second effect, arguing that the above passage constitutes the basis of Ricardo's analysis of exhaustible resources. Then, following Marshall's advice to interpret Ricardo "more generously than he himself interpreted Adam Smith" (Marshall, 1920, p. 813), they conclude that "royalties are there in Ricardo's analysis, but they are not easily identifiable as such" (Kurz & Salvadori, 2009, p. 69), and that although Hotelling rule is not yet to be found in Ricardo, it is not inconsistent with his analysis. In this paper, we set up and begin exploring a "Ricardian" model, where abandoning, exhausting or replanting trees grown on lands of different quality are economic decisions depending on the parameters of the model. Therefore, the model we develop should accommodate both of the effects that Ricardo envisaged. Since a competitive equilibrium with sustained timber production involves a rotation period that is determined by means of the so called "Faustmann formula" (after Faustmann, 1849), which was probably unknown to Ricardo[1], our model is a rational reconstruction of Ricardo's scant remarks on forestry, enriched by the tools of modern theoretical literature on optimal forest management.

We recall that optimal forest management has developed since the late 1970s following Samuelson's review paper (Samuelson, 1976), and mainly focuses on the dynamics of forest rotation on a fully occupied plot of land, where new trees are immediately replanted after old trees are cut (see, e.g. Mitra & Wan, 1985, 1986; Salo & Tahvonen, 2002, 2003; Khan & Piazza, 2012; Fabbri *et al.*, 2015). Immediate replanting and continuous full occupation of the land are justified by the assumptions that cutting and replanting costs are null and that the productive life of trees is finite. Hence such basic model is not suited to handle spatial expansion (or contraction) of forest cultivation: a higher demand for timber simply results in a higher timber price with unchanged production.

To extend the model, we assume positive labour production costs as in Samuelson (1976) and add the Marshallian assumption that the wage rate in terms of the numeraire is given and independent of timber price. Although very specific, this assumption enables a framework in which Marshallian partial equilibrium analysis can be performed. In return, this seems to be the simplest way to shape spatial development without turning the Mitra-Wan model into an explicit multisectoral general equilibrium model.[2]

Making timber a renewable resource that cannot be exhausted, the basic Mitra-Wan model also tends to conceal the fact that optimal forest management implicitly generates dual variables governed by Hotelling-like rules. As pointed out by Salant (2013) in a recent attempt to study the equilibrium price path of timber, this fact is hardly overlooked in settings in which exhaustion is contemplated. Indeed, Salant (2013) used an extreme framework, where replanting costs are infinite, to stress the fact that if a forest is not exploited instantaneously then, at equilibrium, a Hotelling-like rule must hold as the extractor has to be indifferent whether to harvest trees immediately or later. The same kind of phenomena re-emerge in our Ricardian model whenever cutting is cost minimizing but replanting is not viable at the equilibrium prices. However, it will turn out that Hotelling-like rules are relevant in general in equilibrium forest management: indeed, it is competitive arbitrage inducing Hotelling-like asset-market-clearing conditions for aging assets as trees.

In what follows, we concentrate on long run equilibria, making only cursory reference to the more challenging problem of the structure of transitional dynamics, as a proper analytical treatment of our model with non zero production costs, even if limited to the stationary states, can be hardly carried out without a deep study of the dual price system. Therefore, the extension of the basic model has, as a (methodological) side effect, the shift of focus from the quantity side to the price side of the system. According to Salant (2013), this shift is long overdue in literature both for the study of the equilibrium price system – still largely unknown – and as a first step towards the analysis of models in which externalities or other distortions are present[3].

In building our Ricardian model, we borrow the continuous-time production structure of Fabbri *et al.* (2015), instead of using the original discrete-time Mitra-Wan formulation. The continuous-time model is mathematically more challenging than the discrete-time counterpart for two reasons: a) the evolution of state variables is governed by a partial differential equation, as they represent a continuum of (vintage) capital goods: trees of different ages; b) since cost minimization and competitive arbitrage imply that cutting trees is profitable only at a finite set of ages, distributed controls concentrating on single points need be allowed. Hence, in order for the model to be endowed with meaningful price-supported stationary states, intensity levels of the production processes need to be chosen in a very large space (a space of measures). However, once the technical difficulties are overcome (see Fabbri *et al.*, 2015), the clear distinction between stock and flow variables, which is lacking in discrete time (see Foley, 1975), turns into an advantage in the interpretation of the price system: the theory of long run production prices and the role of the Hotelling rule become transparent.

The paper is organized as follows. The continuous-time Ricardian forestry model is introduced in Section 2. In Section 3, modified golden rules of the system are studied. The long run timber supply curve is built in Section 4, where also some comparative static analysis is presented. Section 5 concludes.

2 A Ricardian model of forestry

A number N of lands, with $N \geq 1$, are available for forest cultivation with the purpose of extracting a single final good: timber. The lands have a size given by the coordinates of the positive vector $[h_1, h_2, \cdots, h_N]$, and $x_i(t,s)$ represents the part of land i, $i \in I \equiv \{1, 2, ..., N\}$, covered at time t by trees of a certain age s, with $t \geq 0$, $s \geq 0$. At any time t and for any land i, trees of any age s can be harvested and new saplings can be produced and planted on the land. Let $c_i(t,s)$ be the intensity of cut at time t of trees of age s on land i, and $y_i(t)$ the corresponding rate of production of new saplings. Given an N-tuple of initial distributions $x_i(0,s)$, $i \in I$, the evolution of the system is described by the following set of transport equations and boundary conditions:

$$\begin{cases} \frac{\partial x_i}{\partial t}(t,s) = -\frac{\partial x_i}{\partial s}(t,s) - c_i(t,s) & t > 0, \quad s > 0, \quad i \in I \\ x_i(t,0) = y_i(t) & t \geq 0, \quad i \in I \end{cases} \tag{1}$$

where the variation of density $\frac{\partial x_i}{\partial t}(t,s)$ is due to aging of trees $-\frac{\partial x_i}{\partial s}(t,s)$, and to harvesting $-c_i(t,s)$. Note that the boundary conditions require that the quantities of saplings of age zero planted at time t on the different lands equal the amounts produced $y_i(t)$. In addition, we require the strategy couples $(c_i(t,s), y_i(t))$ to be non negative, that is

$$c_i(t,s) \geq 0, \text{ and } y_i(t) \geq 0, \ \forall t \geq 0, s \geq 0, i \in I, \tag{2}$$

and the trajectories to satisfy the following pure state constraints:

$$\int_0^{+\infty} x_i(t,s)ds \leq h_i, \text{ and } x_i(t,s) \geq 0 \ \forall t \geq 0, s \geq 0, i \in I \tag{3}$$

that is, the occupied portion of the land i equals at most the land extension, and the trees density is nonnegative for all time, ages and lands. Consider now the timber extraction technology. Let $f_i(s)$ be the rate of timber production ensuing from a unitary harvesting ($f_i(s)$ is the *productivity* of a tree of age s on land i) and let $l_i(s)$, $l_i(s) > 0$, be the corresponding unitary cutting cost. Summing up the amounts $f_i(s)c_i(t,s)ds$ of timber extracted from trees of ages s on lands i at time t, we obtain the total timber $q(t)$ harvested at time t

$$q(t) \equiv \sum_{i=1}^{N} q_i(t) \equiv \sum_{i=1}^{N} \int_0^{\infty} c_i(t,s) f_i(s) ds. \tag{4}$$

Similarly, total harvesting costs at time t are given by

$$\sum_{i=1}^{N} \int_0^{\infty} c_i(t,s) l_i(s) ds. \tag{5}$$

Note that costs and productivities are assumed independent from the harvesting rates only for the sake of simplicity. Two additional less innocuous assumptions are instead the following:

(HC1) $l_i(s) \equiv \bar{l}_i$ (unit cutting costs are age-independent);

(HC2) f_i concave, $f_i \geq 0$, $f_i(0) = 0$, and there exists $\bar{s}_i > 0$ such that $f_i(s) = f_i(\bar{s}_i) = f_i^a > 0$ for each $s \geq \bar{s}_i > 0$, and $f_i(s) < f_i^a$ for $s \in [0, \bar{s}_i)$ (in particular, trees not younger than $\bar{s}_i$ are equally productive).

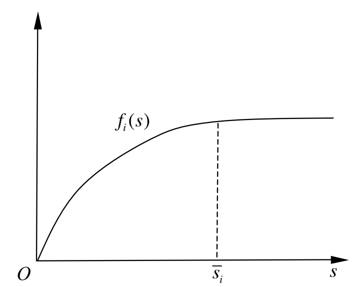

Figure 1: Productivity function.

Note that this implies that $f_i(s)$ is strictly increasing in $[0, \bar{s}_i)$. For the sake of simplicity, we also assume that each f_i is strictly concave in $[0, \bar{s}_i]$ and differentiable in $(0, +\infty)$.[4]

One of the main implications of (HC1) and (HC2) is that, in each land i, trees not younger than $\bar{s}_i$ can be aggregated into a single state variable $a_i(t)$ whose evolution is given by the ordinary differential equation

$$\dot{a}_i(t) = x_i(t, \bar{s}_i) - c_i^a(t), \quad i \in I, \quad t > 0. \tag{6}$$

where

$$a_i(t) = \int_{\bar{s}_i}^{+\infty} x_i(t,s) ds, \text{ and } c_i^a(t) = \int_{\bar{s}_i}^{+\infty} c_i(t,s) ds. \tag{7}$$

Clearly, if $x_i(t, \bar{s}_i) = 0$ for $t \geq 0$, then the wood in old trees on a given land i can be interpreted as the stock of an exhaustible resource in a specific deposit. To complete the description of the technology, we assume linear saplings production costs and write b_i , $b_i > 0$, for the average (and marginal) cost in producing a sapling suitable to be planted on land i. We further assume that planting costs are null *sensu stricto*. Total planting costs at time t are then

$$\sum_{i=1}^{N} y_i(t)b_i. \tag{8}$$

Moreover we assume the demand of timber at price $p_q(t)$, is given by the demand function $D(p_q(t))$, which is taken to be continuous and non-increasing for all $p_q(t) > 0$,[5] and that the interest rate r, $r > 0$, is exogenously given and constant.[6] Since there are constant returns to scale and production is studied under competitive conditions, without loss of generality we may consider a single producer owning all means of production specific of the timber sector and whom, taking the price path of timber as given, maximizes the discounted value of cash flows generated by selling the good and buying the non-specific factors of production. Using (7) both in the objective and in equations (1)–(5), we thus get the following control problem:

$$\text{Max} \int_0^{+\infty} e^{-rt} \left\{ p_q(t)q(t) - \sum_{i=1}^{N} \left[y_i(t)b_i + \bar{l}_i \left(\int_0^{\bar{s}_i} c_i(t,s)ds + c_i^a(t) \right) \right] \right\} \text{d}t \tag{9}$$

$$\begin{cases} \frac{\partial x_i}{\partial t}(t,s) + \frac{\partial x_i}{\partial s}(t,s) = -c_i(t,s) & i \in I, \quad s \in [0,\bar{s}_i], \\ \dot{a}_i(t) = x_i(t,\bar{s}_i) - c_i^a(t) & i \in I, \\ x_i(t,0) = y_i(t) & i \in I, \\ \int_0^{\bar{s}_i} x_i(t,s)ds + a_i(t) \leq h_i & i \in I, \\ c_i(t,s) \geq 0 & i \in I, \quad s \in [0,\bar{s}_i], \\ x_i(t,s) \geq 0 & i \in I, \quad s \in [0,\bar{s}_i], \\ c_i^a(t) \geq 0 & i \in I, \\ a_i(t) \geq 0 & i \in I, \\ y_i(t,s) \geq 0 & i \in I, \\ q(t) = \sum_{i=1}^{N} \left(\int_0^{\bar{s}_i} c_i(t,s)f_i(s)ds + c_i^a(t)f_i^a \right), \end{cases} \tag{10}$$

where the initial conditions $x_i(0,s)$, $s \in [0,\bar{s}_i]$, and $a_i(0)$ for all $i \in I$, and the function $p_q(t)$ are given exogenously. We then define a competitive equilibrium as a price path $p_q(t)$ and a solution of the above control problem such that

$$q(t) = D(p_q(t)). \tag{11}$$

In the next sections, the focus will be the stationary solutions of problem (9)-(10) associated with any given stationary timber price path $p_q(t) = p_q^* \geq 0$. We will then use this analysis to characterize the *long-run timber supply curve* and hence to study the stationary competitive equilibria of the model. Various general remarks are here due. First, note that, except for the properties of the productivity function, the basic Mitra-Wan model in continuous-time in Fabbri *et al.* (2015) is recovered for $b_i = 0$, $l_i(s) = 0$, and for $N = 1$.[7] [8] However, adding different lands in a single final good model (as the one in Fabbri *et al.* (2015)) would not change much the structure of the stationary states. Since candidates for stationary equilibria are the strategy-trajectory components of the modified golden rules and, as for the Ramsey-Cass-Koopmans aggregate model, generically only a unique strategy-trajectory couple can belong to a modified golden rule of the model, the quantity of timber produced in a stationary equilibrium turns out to be independent from the demand of timber (see Mitra & Wan, 1985; Fabbri *et al.*, 2015).[9] In this respect, what makes the extended model a two-final goods system is the introduction of the alternative numeraire good, while the two final goods sectors are interdependent because there are positive production costs on all lands.

Second, an infinite dimensional version of the second welfare theorem would be needed to link the solutions of problem (9)-(10) to competitive equilibria. However, a suitable maximum principle for our problem is technically challenging and at the moment the literature has not yet developed it, although a version of the second welfare theorem restricted to the stationary solutions has been provided by Fabbri *et al.* (2015) in continuous-time. In fact, we will show that a similar results holds for the extended model here discussed. Moreover, since the dual variables supporting a stationary solution of (9)-(10) are themselves stationary, they can be straightforwardly interpreted as the long run production prices that would prevail at the rate of interest r.

Finally, a few preliminary observations on the types of steady states that can solve system (10):

a) a stationary distribution $x_{i^*}(,s)$ solving the state equation (1) is a non increasing function of s, when replanting trees either on a part or on the whole land i^*. An example is provided next.

Example 2.1 Suppose $h_{i^*} = 2$, $\bar{s}_{i^*} = 3$, $c_{i^*}(,s) = \frac{1}{2}$ for $s \in [0,2)$ and $c_{i^*}(,s) = 0$ for $s \in [2,3]$, $a_{i^*} = \frac{1}{2}$ (of course, $c^a_{i^*} = 0$), and $x_{i^*}(,0) = y_{i^*} = 1$. Since also $\int_0^{\bar{s}_{i^*}} c_{i^*}(,s)ds = 1$, that means that the above data are consistent with a stationary $x_{i^*}(,s)$. Then integration of the state equation (1) gives the explicit form of the stationary distribution: $x_{i^*}(,s) = max(0; 1 - \frac{1}{2}s)$. Because in this steady state we have $\int_0^{\bar{s}_{i^*}} x_{i^*}(,s)ds = 1$, half of the land is continuously occupied by an "uneven" cultivated forest. Of what is left, half is abandoned forestland, where old trees stand (i. e. $a_{i^*} = \frac{1}{2}$), and half is bare land. □

b) if a steady state $x_{i^*}(,s)$ is optimal, with a positive aggregate steady state a_{i^*}, then necessarily $x_{i^*}(,s) = 0$ (for instance, the steady state in the previous example cannot be optimal). A formal proof of this result is given in the next section. Indeed, if it is optimal to incur cutting and replanting costs to sustain a stationary forest on part of a land, then it cannot be optimal leaving unextracted timber for which only cutting costs are due. Of course, if the timber extraction costs on land i^* are very high in comparison to timber price, then abandoned forestland is to be expected.

Under these circumstances, the steady state value of the aggregate state variable a_{i^*} will belong to the interval $[0, h_{i^*}]$, while what is left, $h_{i^*} - a_{i^*}$, will be bare land. But if timber extraction is optimal on land i^*, then $a_{i^*} = 0$, and in this case, depending on the demand for timber and on replanting costs, $x_{i^*}(, s)$ will be either positive or zero.
However, even if we had set $a_{i^*} = 0$ in the stationary state of the above example, the resulting stationary state could not have been optimal for a subtle reason: since cutting activities are spread across a whole interval of ages, $x_{i^*}(, s)$ is a strictly decreasing function for $s \in [0, 2]$. On the contrary, it turns out that optimal stationary cutting controls, if positive, are concentrated on a single age. This is the theme of the next section.

3 Modified golden rules

In this section, we concentrate on the modified golden rules of the system. Given a stationary price of timber p_q^*, we formally define a modified golden rule as an N-tuples of couples $[(\bar{c}_i(, s), \bar{c}_i^a, \bar{y}_i, \bar{x}_i(, s), \bar{a}_i), (\bar{p}_i(, s), \bar{p}_i^a, \bar{p}_i^y, \bar{R}_i)]$, $\forall i \in I$, where $(\bar{c}_i(, s), \bar{c}_i^a, \bar{y}_i, \bar{x}_i(, s), \bar{a}_i)$, $s \in [0, \bar{s}_i]$, $\forall i \in I$, is a stationary N-tuples of strategy-trajectory couples that solve the state equations and satisfy the constraints in (10), and where the stationary N-tuples of prices for the different vintages of trees $(\bar{p}_i(, s), \bar{p}_i^a)$, $s \in [0, \bar{s}_i]$, $\forall i \in I$, the stationary N-tuples of sapling prices $\bar{p}_i^y$, $\forall i \in I$, and the stationary N-tuples of rent rates $\bar{R}_i$, $\forall i \in I$, are such that, at the given prices: (A) profits are maximized, (B) the markets for lands' services clear, (C) the asset-market-clearing conditions that hold under competitive arbitrage are satisfied. The focus is on modified golden rules because it turns out that their strategy-trajectory components are optimal solutions of problem (9)–(10) and, conversely, that any optimal stationary solution of problem (9)–(10) can be endowed with a stationary supporting price function.[10]
Consider now the implications of the above conditions (A), (B) and (C). First, observe that profit maximizarion and constant returns to scale imply null maximum profits both in cutting trees and in saplings production. Hence, the following inequalities must hold for all $i \in I$:

$$f_i(s)p_q^* \leq \bar{p}_i(, s) + \bar{l}_i, \quad \bar{c}_i(, s)f_i(s)p_q^* = \bar{c}_i(, s)(\bar{p}_i(, s) + \bar{l}_i), \quad s \in [0, \bar{s}_i] \tag{12}$$

$$f_i^a p_q^* \leq \bar{p}_i^a + \bar{l}_i, \quad \bar{c}_i^a f_i^a p_q^* = \bar{c}_i^a(\bar{p}_i^a + \bar{l}_i), \quad \bar{c}_i^a \geq 0, \tag{13}$$

$$\bar{p}_i^y \leq b_i, \quad \bar{y}_i \bar{p}_i^y = \bar{y}_i b_i, \quad \bar{y}_i \geq 0, \tag{14}$$

$$\bar{c}_i(, s) \geq 0, \quad s \in [0, \bar{s}_i]; \tag{15}$$

where, in particular, the meaning of conditions (12) (13) and (15) is that no cutting process generates extra profits and that only processes with zero losses can be activated, whereas conditions (13) and (14) imply that sapling production can occur only if costs are covered.

Second, since lands are supplied inelastically, requirement (B) is satisfied if and only if the following conditions hold for all $i \in I$:

$$\int_0^{\bar{s}_i} \bar{x}_i(,s)ds + \bar{a}_i \leq h_i \tag{16}$$

$$\bar{R}_i[\int_0^{\bar{s}_i} \bar{x}_i(,s)ds + \bar{a}_i] = \bar{R}_i h_i \tag{17}$$

$$\bar{R}_i \geq 0. \tag{18}$$

Finally, we note that standing trees are exhaustible resources, so the asset-market-clearing conditions must be instances of the Hotelling rule, even if, differently from the standard case, two specific facts affect the precise structure of the price equations: (1) a rent rate is due to hold a tree of any age *in situ*, (2) young trees are subject to aging. Fact (1) implies that for each i the price of mature trees evolves according to

$$\dot{p}_i^a(t) \leq rp_i^a(t) + R_i(t), \quad a_i(t)\dot{p}_i^a(t) = a_i(t)[rp_i^a(t) + R_i(t)]. \tag{19}$$

Hence, for the price of old trees on land i to be stationary at least one of the following systems must hold:[11]

$$0 = r\bar{p}_i^a + \bar{R}_i, \quad \bar{a}_i \geq 0, \tag{20}$$

or

$$0 \leq r\bar{p}_i^a + \bar{R}_i, \quad \bar{a}_i = 0. \tag{21}$$

On the other hand, fact (2) implies that if prices of young trees on different lands are stationary, then Hotelling rule holds *across* ages.[12] Hence, for all $i \in I$ we have the following conditions:

$$\frac{d\bar{p}_i}{ds}(,s) \leq r\bar{p}_i(,s) + \bar{R}_i \quad s \in [0, \bar{s}_i] \tag{22}$$

$$\bar{x}_i(,s)\frac{d\bar{p}_i}{ds}(,s) = \bar{x}_i(,s)[r\bar{p}_i(,s) + \bar{R}_i] \quad s \in [0, \bar{s}_i] \tag{23}$$

$$\bar{x}_i(,s) \geq 0 \quad s \in [0, \bar{s}_i]. \tag{24}$$

Competitive arbitrage has a further implication: the price of a tree cannot jump up at junction points (otherwise there would be a rush to buy the asset just before it appreciates), while jumsp down can occur only if no agent holds the asset (otherwise there would be a rush to sell the depreciating tree). This implies

$$\bar{p}_i(,0) \leq \bar{p}_i^y, \quad \bar{x}_i(,0)\bar{p}_i(,0) = \bar{x}_i(,0)\bar{p}_i^y, \tag{25}$$

and

$$\bar{p}_i(,\bar{s}) \geq \bar{p}_i^a, \quad \bar{x}_i(,\bar{s})\bar{p}_i(,\bar{s}) = \bar{x}_i(,\bar{s})\bar{p}_i^a, \tag{26}$$

for all $i \in I$. We now identify the modified golden rules of our Ricardian model. Since the system comprising the stationary versions of (10) and of the supporting price conditions (12)–(26) can be split into N independent systems, each referring to a single land, we state the results for a generic land i.We start from what anticipated in section 2.

Proposition 3.1 *Assume $(\bar{x}_i(,s), \bar{a}_i)$ belong to a modified golden rule. Then:*

(*i*) $f_i^a p_q^* > \bar{l}_i \Longrightarrow \bar{a}_i = 0$.

(*ii*) $f_i^a p_q^* \leq \bar{l}_i \Longrightarrow \bar{x}_i(,s) = 0 \quad \forall s \in [0, \bar{s}_i]$.

Moreover, if $f_i^a p_q^* \leq \bar{l}_i$, then the strategy-trajectory couple $\bar{c}_i(,s) = 0 \quad \forall s \in [0, \bar{s}_i]$, $\bar{c}_i^a = \bar{y}_i = 0$, $\bar{x}_i(,s) = 0 \quad \forall s \in [0, \bar{s}_i]$, $\bar{a}_i \in [0, h_i]$, and the price system $\bar{p}_i(,s) = 0 \quad \forall s \in [0, \bar{s}_i]$, $\bar{p}_i^y = \bar{p}_i^a = \bar{R}_i = 0$ constitute a modified golden rule.

Proof. For (*i*), note that $f_i^a p_q^* > \bar{l}_i$ and inequality (13) imply $\bar{p}_i^a > 0$, so (20) cannot hold and (21) holds instead. To prove (*ii*), assume by contradiction that $f_i^a p_q^* \leq \bar{l}_i$ and there exists an age s such that $\bar{x}_i(,s) > 0$. Since there is planting, i.e. $\bar{y}_i > 0$, then (14), and (25) imply $\bar{p}_i(,0) = b_i$. Moreover, since stationary paths are non increasing, there exists a maximum age $\hat{s}$ of standing trees on $[0, \bar{s}_i]$, more precisely, $\hat{s} \equiv \sup\{s \in [0, \bar{s}_i] \ : \ \bar{x}_i(,s) > 0\}$. Then (22) holds with equality up to age $\hat{s}$, implying

$$\bar{p}_i(,s) = b_i e^{rs} + \frac{\bar{R}_i}{r}(e^{rs} - 1), \quad s \in [0, \hat{s}] \tag{27}$$

so that $\bar{p}_i(,s) > 0$ for all $s \in [0, \hat{s}]$. Now, the productivity function is increasing from zero to maturity age, so that

$$f_i(s) p_q^* \leq f_i^a p_q^* < \bar{p}_i(,s) + \bar{l}_i \quad \forall s \in [0, \bar{s}_i].$$

From (12) one derives that no standing tree can be cut, implying $\dot{a}_i(t) > 0$, and hence a contradiction. Direct substitution of the candidate modified golden rule into inequalities (12)–(18), (20) and (26), with $\frac{\partial x_i}{\partial t}(t,s) = 0$ and $\dot{a}_i(t) = 0$ into system (10), gives the last claim. □

Remark 3.2 Interpreting p_q^* as the choke off price at which timber demand is nil, Proposition 3.1 (*i*) is the standard result in theory of exhaustible resources that exhaustion of a deposit is optimal if the choke off price is higher than the extraction cost. □

It remains to be determined if forest cultivation takes place in the modified golden rules when $f_i^a p_q^* > \bar{l}_i$, and to find which technique is chosen if this occurs. If it does, proceeding as in the proof of Proposition 3.1 (*ii*), one derives $\bar{p}_i(,s)$ given by (27). That, substituted into the no-extra profits condition (12) and rearranging terms, provides

$$\frac{f_i(s) p_q^* - b_i e^{rs} - \bar{l}_i}{e^{rs} - 1} \leq \frac{\bar{R}_i}{r} \quad s \in [0, \bar{s}_i). \tag{28}$$

Note also that, using (13) and (26), this last condition can be extended to the closed interval $[0, \bar{s}_i]$. So in the end, if forest cultivation takes place in a modified golden rule, (28) need to be verified everywhere, and it has to hold with equality for at least an age $\hat{s}$ in the interval $[0, \bar{s}_i]$. Of course, whenever timber price is too low, a negative rent would be required to verify this condition, implying that forest cultivation is not profitable at that price.

This line of argument leads to the basic problem of the choice of technique, that is, finding the ages at which the land value $\bar{R}_i/r$ attains a minimum in the set of values that satisfy (28) or, equivalently, the ages that solve the Faustmann problem of maximizing "the present discounted value of all net cash receipts [...] calculated over the *infinite chain* of cycles of planting on the given acre of land from now until Kingdom Come" (Samuelson, 1976, p. 122), namely

$$V_i^F(s) = \sum_{n=1}^{\infty} e^{-rns}[f_i(s)p_q^* - b_i e^{rs} - \bar{l}_i] = \frac{f_i(s)p_q^* - b_i e^{rs} - \bar{l}_i}{e^{rs} - 1}. \tag{29}$$

This requirement, emerging in different forms in all forestry management problems where replanting is possible, is used also here to characterize modified golden rules for timber prices greater than $\bar{l}_i/f_i^a$.

To begin with, since $V_i^F(s)$ is continuous in the interval $s \in [\lambda, \bar{s}_i]$ for all $0 < \lambda \leq \bar{s}_i$, and $V_i^F(s) \to -\infty$ for $s \to 0_+$, the Faustmann problem has a solution for each $p_q^* \geq 0$. Moreover, the maximum value as a function of the timber price $M_i^F(p_q^*)$ is negative for low values of the price p_q^*, increasing with p_q^*, and eventually positive when p_q^* is sufficiently high. Therefore, we can define the "break-through" price $\hat{p}_{qi}^*$ as the minimum timber price for which the maximum of the Faustmann function is non-negative.

Now note that $\hat{p}_{qi}^*$ is strictly greater than $\bar{l}_i/f_i^a$: for timber price levels that are only slightly higher than $\bar{l}_i/f_i^a$, even if cutting costs can be covered by waiting for new planted trees to reach maturity, forest cultivation still results in losses, due to strictly positive planting costs. Hence, for all $p_q^* \in (\bar{l}_i/f_i^a, \hat{p}_{qi}^*)$

$$\max_{s\in[0,\bar{s}_i]}[f_i(s)p_q^* - b_i e^{rs} - \bar{l}_i] < 0,$$

and it is easy to see that there exists $m_i(p_q^*)$, with $0 < m_i(p_q^*) < b_i$, such that

$$\max_{s\in[0,\bar{s}_i]}[f_i(s)p_q^* - m_i(p_q^*)e^{rs} - \bar{l}_i] = 0.$$

With this facts established, we are ready to characterize modified golden rules for $p_q^* \in (\bar{l}_i/f_i^a, \hat{p}_{qi}^*)$.

Proposition 3.3 *Assume $p_q^* \in (\frac{\bar{l}_i}{f_i^a}, \hat{p}_{qi}^*)$, and that $(\bar{x}_i(,s), \bar{a}_i)$ belong to a modified golden rule. Then:*

(i) $\bar{x}_i(,s) = 0$, $\forall s \in [0, \bar{s}_i]$, *and* $\bar{R}_i = 0$.

(ii) *The strategy-trajectory couple $\bar{c}_i(,s) = 0$, for all $s \in [0, \bar{s}_i]$, $\bar{c}_i^a = \bar{y}_i = 0$, $\bar{x}_i(,s) = 0$, for all $s \in [0, \bar{s}_i]$, $\bar{a}_i = 0$, and the price system $\bar{p}_i(,s) = m_i(p_q^*)e^{rs}$, for all $s \in [0, \bar{s}_i]$, $\bar{p}_i^y = m_i(p_q^*)$, $\bar{p}_i^a = m_i(p_q^*)e^{r\bar{s}_i}$, $\bar{R}_i = 0$ constitute a modified golden rule.*

Proof. To prove (i), one may replicate the argument used to establish Proposition (3.1) (ii). Assume by contradiction $\bar{x}_i(,s) > 0$ at an age s in $[0, \bar{s}_i]$, then both planting and cutting at some age have to occur. Hence there exists $\check{s} \in [0, \bar{s}_i]$ such that (28) holds as an equality at $s = \check{s}$. Since by assumption $V_i^F(\check{s}) < 0$, this contradicts the non-negativity of the rent rate. Hence $\bar{x}_i(,s) = 0$, for all $s \in [0, \bar{s}_i]$. Moreover, since $p_q^* > \bar{l}_i$ implies $\bar{a}_i = 0$, it is immediate from (16) and (17) that $\bar{R}_i = 0$.
To prove (ii), note that the strategy-trajectory couple in the candidate modified golden rule is a stationary solution of system (10) and that inequalities (12)–(18) and (21) are verified by the candidate price system. Finally, note that for our candidate modified golden rule inequalities (22), (25) and (26) hold as equalities. □

The next two propositions show that forest cultivation becomes profitable for $p_q^* \geq \hat{p}_{qi}^*$ and describe modified golden rules when $p_q^* = \hat{p}_{qi}^*$ and when $p_q^* > \hat{p}_{qi}^*$, respectively. But first, consider the Faustmann problem for $p_q^* \geq \hat{p}_{qi}^*$. A key fact is that the maximizer $s_i^F(p_q^*)$ is unique.[13] This is quite apparent from Figure 2, where we drawed the graph of the function

$$g_i(s) := \frac{M_i^F(p_q^*)(e^{rs} - 1) + b_i e^{rs} + \bar{l}_i}{p_q^*} \tag{30}$$

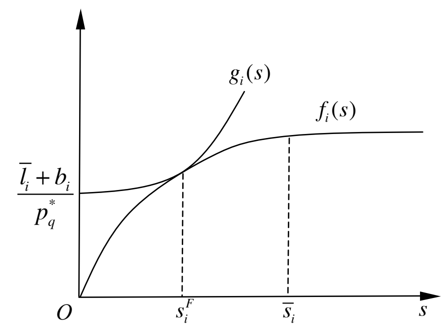

Figure 2: Uniqueness of $s_i^F(p_q^*)$.

along with the graph of the productivity function $f_i(s)$. Since $M_i^F(p_q^*) \geq 0$, $g_i(s)$ is a strictly convex increasing function, and this, together with the concavity of $f_i(s)$, implies uniqueness of the age that maximizes the Faustmann function (29). The implication of this fact is, as we will show, that forest cultivation in modified golden rules is characterized by a uniform density function on the cultivated land, with cutting concentrated at the Faustmann age. Formally, we will consider the uniform density functions given by

$$x_{\theta_i}^{U_i}(,s) := \frac{\theta_i h_i}{s_i^F(p_q^*)} \chi_{[0, s_i^F(p_q^*)]}(s), \tag{31}$$

where $\theta_i \in [0,1]$ is the share of land i that is cultivated, in which all ages in the range $[0, s_i^F(p_q^*)]$ are uniformly distributed and equal to $\frac{\theta_i h_i}{s_i^F(p_q^*)}$, while those in the range $[s_i^F(p_q^*), \bar{s}_i]$ are null, and the cutting intensity vectors given by

$$c_{\theta_i}^{D_i}(,s) \equiv \frac{\theta_i h_i}{s_i^F(p_q^*)} \delta_{s_i^F(p_q^*)}, \tag{32}$$

where $\delta_{s_i^F(p_q^*)}$ is the Dirac Delta at point $s_i^F(p_q^*)$, that is, the action undertaken by $c_{\theta_i}^{D_i}(,s)$ is cutting exactly the trees reaching age $s_i^F(p_q^*)$. Note that if $\theta_i > 0$, then $c_{\theta_i}^{D_i}(,s)$ is not a function of s but a positive measure.

Proposition 3.4 *Assume $p_q^* = \hat{p}_{qi}^*$. Then:*

(*i*) $\bar{R}_i = 0$ *for any modified golden rule. In addition, a strategy-trajectory couple belongs to a modified golden rule only if it has the following form: $\bar{c}_i^a = \bar{a}_i = 0$, $\bar{x}_i(,s) = x_{\theta_i}^{U_i}(,s)$, $\bar{c}_i(,s) = c_{\theta_i}^{D_i}(,s)$ and $\bar{y}_i = \frac{\theta_i h_i}{s_i^F(\hat{p}_{qi}^*)}$, for any choice of $\theta_i \in [0,1]$. Moreover, if $\theta_i > 0$, then $\bar{p}_i^y = b_i$, $\bar{p}_i(,s) = b_i e^{rs}$, for all $s \in [0, s_i^F(\hat{p}_{qi}^*)]$.*

(*ii*) *The strategy-trajectory couple $\bar{x}_i(,s) = x_{\theta_i}^{U_i}(,s)$, $\bar{c}_i(,s) = c_{\theta_i}^{D_i}(,s)$ and $\bar{y}_i = \frac{\theta_i h_i}{s_i^F(p_q^*)}$, any $\theta_i \in [0,1]$, $\bar{c}_i^a = \bar{a}_i = 0$, and the price system $\bar{p}_i(,s) = b_i e^{rs}$, for all $s \in [0, \bar{s}_i]$, $\bar{p}_i^y = b_i$, $\bar{p}_i^a = b_i e^{r\bar{s}_i}$, $\bar{R}_i = 0$ constitutes a modified golden rule.*

Proof. To prove (i), consider any modified golden rule. It is immediate from $\hat{p}_{qi}^* > \bar{l}_i$ that $\bar{c}_i^a = 0$ and $\bar{a}_i = 0$ hold. Then, if no planting occurs, $\bar{R}_i = 0$, as in Proposition 3.3. On the other hand, if $\bar{x}_i(,s) > 0$ for at least an age s in $[0, \bar{s}_i]$, then there exists $\check{s} \in [0, \bar{s}_i]$ such that (28) holds as an equality at $s = \check{s}$ and, since $V_i^F(s) < 0$ for all $s \in [0, \bar{s}_i] \setminus \{s_i^F(\hat{p}_{qi}^*)\}$ while $V_i^F(s_i^F(\hat{p}_{qi}^*)) = 0$, this implies that $\check{s} = s_i^F(\hat{p}_{qi}^*)$ and $\bar{R}_i = 0$. When $\theta_i > 0$, $\bar{p}_i^y = b_i$ follows from (14), and $\bar{p}_i(,s) = b_i e^{rs}$, for all $s \in [0, s_i^F(\hat{p}_{qi}^*)]$ from the fact that on this set of ages (22) holds as an equality. Finally, (ii) is verified by direct substitution. □

The following Proposition shows that the multiplicity of modified golden rules arising for $p_q^* = \hat{p}_{qi}^*$ disappears for higher levels of timber price.

Proposition 3.5 *Assume $p_q^* > \hat{p}_{qi}^*$. Then:*

(*i*) *For any modified golden rule, $\bar{R}_i = rM_i^F(p_q^*)$, $\bar{x}_i(,s) = x_1^{U_i}(,s)$, $\bar{c}_i(,s) = c_1^{D_i}(,s)$, $\bar{y}_i = \frac{h_i}{s_i^F(p_q^*)}$, $\bar{p}_i^y = b_i$, and $\bar{p}_i(,s) = b_i e^{rs} + M_i^F(p_q^*)(e^{rs} - 1)$, for all $s \in [0, s_i^F(p_q^*)]$.*

(*ii*) *The strategy-trajectory couple $\bar{x}_i(,s) = x_1^{U_i}(,s)$, $\bar{c}_i(,s) = c_1^{D_i}(,s)$, $\bar{y}_i = \frac{h_i}{s_i^F(p_q^*)}$, $\bar{c}_i^a = \bar{a}_i = 0$, and the price system $p_i(,s) = b_i e^{rs} + M_i^F(p_q^*)(e^{rs} - 1)$, for all $s \in [0, \bar{s}_i]$, $\bar{p}_i^y = b_i$, $\bar{p}_i^a = b_i e^{r\bar{s}_i} + M_i^F(p_q^*)(e^{r\bar{s}} - 1)$, $\bar{R}_i = rM_i^F(p_q^*)$ constitutes a modified golden rule.*

Proof. After noting that $\bar{R}_i < rM_i^F(p_q^*)$ is inconsistent with the no extra profits condition (12) and that forest cultivation results in losses whenever $\bar{R}_i > rM_i^F(p_q^*)$, we can proceed as in the proof of Proposition 3.4, taking into account that $\bar{R}_i > 0$ implies that (16) holds with equality. □

Having characterized modified golden rules for the different values of the timber price, we have also implicitly derived the "long run timber supply correspondence", defined by equation (4) when cutting intensity levels belong to modified golden rules. We can now turn our attention to the properties of the long run timber supply curve and to the analysis of the competitive stationary equilibrium of our forestry model.

4 Long run supply curves and comparative statics effects of an increase in timber demand

As ordinary supply functions (or correspondences) are the sum of individual firms supply functions, our aggregate timber supply correspondence is simply the horizontal summation of the single lands supply correspondences. Thus our first task is the construction of land i timber supply:

- For $0 \leq p_q^* < \hat{p}_{qi}^*$, timber supply on land i is constant at zero. Indeed Proposition 3.1 (i) implies $\bar{c}_i^a f_i^a = 0$ for all $p_q^* \geq 0$, and (ii) of the same proposition and Proposition 3.3 (i) imply $\int_0^{\bar{s}_i} \bar{c}_i(,s) f_i(s) ds = 0$;

- However, when $p_q^* > \bar{l}_i / f_i^a$ something economically relevant happens under the surface of the constant supply function. Since now it is worth extracting timber they contain, the price of old trees is not anymore zero. Old trees have become a valuable resource, destined to be exhausted.

- When $p_q^* = \hat{p}_{qi}^*$, the supply curve has a flat. This follows from Proposition 3.4 (i) and the definition of the Dirac delta, that is $\delta_{s_i^F(\hat{p}_{qi}^*)} f_i(s) = f_i(s_i^F(\hat{p}_{qi}^*))$, so that
$$\int_0^{\bar{s}_i} c_{\theta_i}^{D_i}(,s) f_i(s) ds = \theta_i \frac{h_i f_i(s_i^F(\hat{p}_{qi}^*))}{s_i^F(\hat{p}_{qi}^*)}, \quad \text{for any } \theta_i \in [0,1].$$
The amount of timber supplied is any quantity in the interval $[0, h_i f_i(s_i^F(\hat{p}_{qi}^*))/s_i^F(\hat{p}_{qi}^*)]$.

- Finally, for $p_q^* > \hat{p}_{qi}^*$ we can use Proposition (i) 3.5 to get
$$\int_0^{\bar{s}_i} c_1^{D_i}(,s) f_i(s) ds = \frac{h_i f_i(s_i^F(p_q^*))}{s_i^F(p_q^*)},$$
and hence to establish that the supply correspondence is univalued. In addition, the supply function is increasing at all $p_q^* > \hat{p}_{qi}^*$. To prove that, we show that the Faustmann critical age $s_i^F(p_q^*)$ is decreasing at any price greater than $\hat{p}_{qi}^*$.

Proposition 4.1 *Assume $p_q^* \geq \hat{p}_{qi}^*$. Then $s_i^F(p_q^*)$ is a continuous decreasing function. As a consequence, land i supply function $h_i f_i(s_i^F(p_q^*))/s_i^F(p_q^*)$ is continuous and increasing for all $p_q^* > \hat{p}_{qi}^*$.*

Proof. Let p_q^* be any price that satisfies the hypothesis $p_q^* \geq \hat{p}_{qi}^*$ and let $\Delta p_q^* > 0$. Recall the definition of $g_i(s)$ in (30), and define
$$k_i(s) := \frac{M_i^F(p_q^* + \Delta p_q^*)(e^{rs} - 1) + b_i e^{rs} + \bar{l}_i}{p_q^* + \Delta p_q^*}, \quad \alpha := \frac{(p_q^* + \Delta p_q^*)(M_i^F(p_q^*) + b_i)}{p_q^*(M_i^F(p_q^* + \Delta p_q^*) + b_i)}.$$

Note that the following hold: $g_i(0) > k_i(0)$, and $g'_i(s) = \alpha k'_i(s)$. Now, since $\alpha \geq 1$ would lead to the contradiction $k_i(s_i^F(p_q^*)) < g_i(s_i^F(p_q^*)) = f_i(s_i^F(p_q^*)$, $\alpha < 1$. Hence, at $s_i^F(p_q^* + \Delta p_q^*)$ the following hold true: $k'_i(s_i^F(p_q^* + \Delta p_q^*)) = f'_i(s_i^F(p_q^* + \Delta p_q^*)) > g'_i(s_i^F(p_q^* + \Delta p_q^*))$. Since we have $f'_i(s_i^F(p_q^*)) = g'_i(s_i^F(p_q^*))$ and since $f_i(s) - g_i(s)$ is a strictly concave function, the last inequality implies $s_i^F(p_q^* + \Delta p_q^*) < s_i^F(p_q^*)$. Finally, uniqueness of the maximizer of the Faustmann function implies that $s_i^F(p_q^* + \Delta p_q^*) \to s_i^F(p_q^*)$ for $\Delta p_q^* \to 0$, and hence the continuity of the function $s_i^F(p_q^*)$. the last fact is a direct consequence of (HC2), since $h_i f_i(s)/s$ is decreasing in s. □

This completes the construction of the supply correspondence on land i for all $p_q^* \geq 0$. We have depicted a typical supply curve in Figure 3.

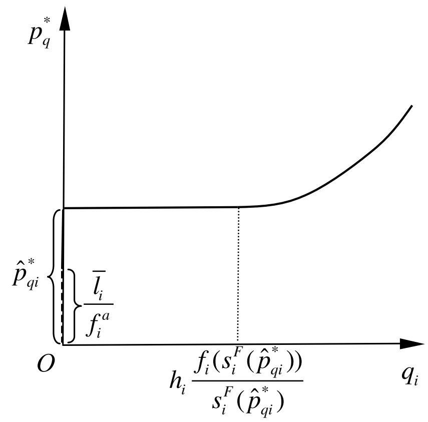

Figure 3: Supply curve.

Now we can construct the aggregate timber supply correspondence that, as already noted, is simply the horizontal sum of the N individual correspondences. If, to avoid singular cases, we assume $\hat{p}_{qi}^* \neq \hat{p}_{qj}^*$ and $(\bar{l}_i/f_i^a) \neq (\bar{l}_j/f_j^a)$, for all $i, j \in I, i \neq j$, then the aggregate supply curve contains exactly N flats in correspondence of the break through prices $\hat{p}_{qi}^*$ and N threshold prices $\bar{l}_i/f_i^a$ that trigger extraction of the timber contained in old trees.

The construction of the curve is illustrated in Figure 4 for the case $N = 2$.

Note that through this process lands are ranked both in terms of their threshold prices $\bar{l}_i/f_i^a$ and in terms of their break through prices $\hat{p}_{qi}^*$. The order in terms of break through prices, which is called the *order of fertility* (see Kurz & Salvadori, 1995, p. 287), is one of the building blocks of the Ricardian theory of extensive rent, and gives the order in which the lands are taken into cultivation when the demand for timber increases. Timber production begins when the price of timber equals the minimum $\hat{p}_{qi}^*$, that is the break through price of the most fertile land ($\hat{p}_{q1}^*$ in the example in Figure 4). Since at this price any share of the most fertile land can be cultivated, the supply curve has a flat. Before passing to the second most fertile land (that in the example in Figure 4 occurs when the price $\hat{p}_{q2}^*$ reached), forest cultivation on the first land is intensified (recall that the Faustmann age of the cost minimizing technique is a decreasing function of the price of timber), with the effect that timber supply increases and a *Faustmann intensive rent* is paid on the most fertile land (see Sraffa, 1925, p. 334 ss., for a similar construction).

When the break through price of the second most fertile land is reached ($\hat{p}_{q2}^*$ in Figure 4), there is a second flat on the supply curve. If not fully cultivated, the second most fertile land is now the *Ricardian marginal land* on which no rent is paid, so that the rent paid on the most fertile land is now the usual *Ricardian extensive rent* that eliminates the extra profits that the use of the cost-minimizing technique on the most fertile land would otherwise generate. For higher levels of the timber price, cultivation is intensified on both the first two most fertile lands until the break through price of the third most fertile land is reached, where there is a new flat in the supply curve. And so on.

On the other hand, and quite naturally, the order in terms of threshold prices, which we call the *order of extraction*,[14] does not affect the shape of the long run aggregate supply correspondence, although it determines the structure of the state variable in the modified golden rules that are behind the supply curve. We will say that there is an *order of the lands* if each land occupies the same position in the two orders described above. An order of lands will be called *strong* if, in addition, $\hat{p}^*_{qi} < \bar{l}_j/f^a_j$ for each land j that follows land i in the order of lands (for example, that depicted in Figure 4 is a strong order of lands).

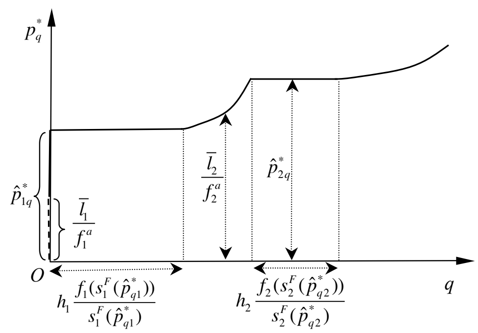

Figure 4: Aggregate supply.

Once the properties of the supply correspondence have been established, the comparative statics effects of an increased demand for timber are fairly obvious: if at the initial equilibrium the marginal land exists, then a higher demand could simply lead to an increase of the timber produced on that marginal land, without any increase in timber price and in rents of the intra-marginal lands. If on the contrary the lands in use are fully occupied, then a rise in the price of timber and in the rents of the fully occupied lands necessarily occurs. As suggested by Ricardo, in both cases a sufficiently high demand for timber causes a "rent to be paid for forests [...], which could before afford no rent", as future demand cannot be met without fully cultivating some of previously unoccupied or partially cultivated lands.

The other part of Ricardo's argument – that not all of what "is annually paid by a farmer to his landlord" can be considered rent, and that a higher demand for timber could simply raise those components of the compensation paid by the farmer that are not rent – has the clearest counterpart in our model when there is a strong order of lands. Figure 5 illustrates the point. Let the initial demand curve be the one labeled A in the figure. Then, since in equilibrium the timber price is lower than $\bar{l}_2/f^a_2$, all trees standing on land 2 have no value. If an increase in demand shifts the curve to $B3$, then both the prices of standing trees and the rent on land 2 become positive, but if instead the increase is such that the demand curve is either $B1$ or $B2$, then no rent will be paid on land 2, although in both cases the owners of land 2 will receive a compensation whenever they sell the now valuable assets standing on their land.

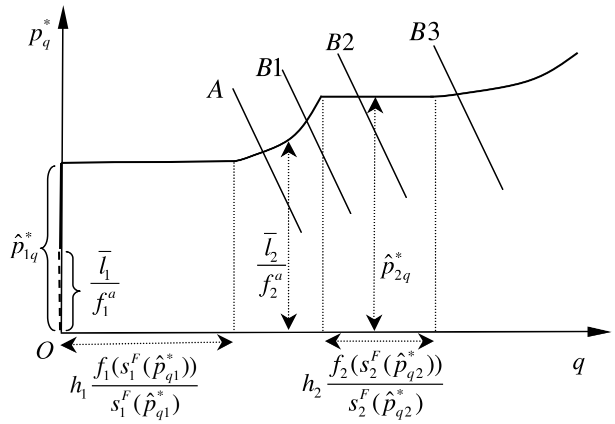

Figure 5: Effects of an increasing timber price.

5 Concluding remarks

We have provided a rational basis for Ricardo's tenet that the compensations received by the owners of marginal forestlands in Norway following the raise of timber demand (and hence of timber price) in more southern countries of Europe were not rents. To do that, relying on the insights in Kurz & Salvadori (2009, 2011) and on the continuous-time forestry management theory developed in Fabbri *et al.* (2015), we have built a "Ricardian" forestry model whose comparative statics properties, we think, well mimics Ricardo's claims.

Beyond its use as an interpretive tool, our model improves that contained in Fabbri *et al.* (2015) by introducing, although in a simplified way, a second sector in the continuous-time Mitra-Wan forestry framework. Starting from this basis, it should be no hard task to develop alternative two-sector models with, for example, a second agricultural sector or an energy sector that uses both fossil fuels and renewable energy technologies. With these modifications, the model can be adapted to the analysis of important contemporary economic issues as, for example, the following ones (e.g. Piazza & Roy, 2015). Is it truly inevitable that forestry or trees conservation will be adversely affected by rising food demand or the introduction of policies encouraging the use of biofuels?

Notes

[1]However, a recent study has shown that the formula was already known in England at the end of 17th century and that its discovery is linked to the institutional developments that changed the English financial markets after the 1690s; see Viitala (2013).

[2]Using a Sraffian terminology, we are treating timber as a non-basic commodity.

[3] Stokey and Lucas, 1989, early advocated the direct study of the equilibrium price system, to supplement what they called "the indirect approach".

[4] Besides Ricardian extensive rents, an intensive rent arises in the model because a higher timber price leads to a shorter rotation period, with higher timber production and higher rent, on fully occupied lands. We call this type of intensive rent the *Faustmann rent*.

[5] Note that a perfectly elastic timber demand curve would not satisfy the requirements in the text, as it would be multivalued. We avoid this "linear" case because it complicates the comparative statics. It should be noted that, in the continuous-time version of the basic Mitra-Wan model, the closed form of the dynamics of competitive equilibrium is known only the linear case (see Fabbri *et al.*, 2015).

[6] Although the case $r = 0$ is important in forestry literature (price-supported steady states of the original undiscounted Mitra-Wan model fulfill the foresters' goal of *maximum sustained yield*), its analysis would require the introduction of specific optimality criteria. However, on a first inspection, it seems some of the results in Fabbri *et al.* (2015) may extend to the model discussed in this paper.

[7] In Fabbri *et al.* (2015), $f(s)$ is not required to be concave, has support contained in $(0, \bar{s})$, with $0 < \bar{s} < \infty$, meaning that trees older than $\bar{s}$ are considered unproductive and that some time after planting is needed before a tree becomes productive. Moreover, since cutting/replanting costs are null, Fabbri *et al.* (2015) assume consistently that the aggregate variables $a(t)$ and $c_a(t)$ are zero at any time.

[8] Fabbri *et al.* (2015) consider an optimal growth model with the Ramsey-like objective $\int_0^{+\infty} e^{-\rho t} u\left(q^D(t)\right)\, \mathrm{d}t$, where $\rho \geq 0$ is the rate of discount and $u(q^D(t))$ is the instantaneous utility function. To compare the two models, it is sufficient to re-interpreted the rate of interest as the rate of discount and the demand function $q^D(t) = D(p_q(t))$ as the inverse of the function $p_q = u'(q^D(t))$.

[9] Following the terminology of optimal growth theory, we call *modified golden rule* any stationary strategy-trajectory couple satisfying (10) and supported by a stationary price path. A formal definition is given in section 3.

[10] The proofs of these results, that are not given here, can be obtained by adapting Theorem 4.5 in Fabbri *et al.* (2015).

[11] In principle, negative prices are possible in this model. For example, suppose that on land i the cost of removal of the trees is very high, so that $f_i^a p_q^* - \bar{l}_i < 0$, and $a_i(0) = h_i$. In this case, a stationary state in which $\bar{a}_i = h_i$ can be sustained by negative prices of mature trees and positive rent rates that satisfy $f_i^a p_q^* - \bar{l}_i \leq \bar{p}_i^a$ and $0 = r\bar{p}_i^a + \bar{R}_i$. Note that in this kind of equilibria, even if the rent rate is positive, what is "annually paid by a farmer to his landlord" equals zero because a positive rent exactly compensates for the interests the landlord pays on the value of the "bad" standing on his land. However this case is not particularly relevant as, besides a stationary solution with negative price of the old trees, there exists also the more natural non-negative solution $\bar{R}_i = 0$, $\bar{p}_i^a = 0$. Had been the land short in supplying timber to production, then a more interesting case of negative price for mature trees would have occurred: a high removal cost on mature forestland and a positive rent rate would result in an unavoidable negative price for old trees. However, our assumptions (HC1)-(HC2) preclude this possibility, implying that the rent on land i can be positive only if $f_i^a p_q^* - \bar{l}_i > 0$.

[12] A simple argument that explains the statement in the text runs as follows. Buying at time t a tree of age s on land i, holding it *in situ* till time $t + \Delta t$, and then selling it generates a net revenue at time $t + \Delta t$ of $p_i(t + \Delta t, s + \Delta t) - p_i(t, s) - \bar{R}_i \Delta t$. Under competitive arbitrage, this sum equates the foregone interest on the sum used to buy the asset, $rp_i(t, s)\Delta t$. Hence:

$$\frac{[p_i(t + \Delta t, s + \Delta t) - p_i(t, s + \Delta t)] + [p_i(t, s + \Delta t) - p_i(t, s)]}{\Delta t} = rp_i(t, s) + \bar{R}_i.$$

With stationary prices, letting $\Delta t \to 0_+$ one gets the Hotelling-like condition in the text.

[13] Uniqueness can be proved for all $p_q^* \geq 0$, however irrelevant when $p_q^* < \hat{p}_{qi}^*$.

[14]Although our analysis is static, we abuse the term *order of extraction* to underline the fact that if planting costs were infinite and all standing trees were old, then our model would be reduced to a standard exhaustible resource model with multiple deposits. Provided the demand choke off price is greater that the maximum of the $\bar{l}_i/f_i^a$, $i \in I$, the order in term of threshold prices in this model would correspond to the order of extraction of the N deposits of timber (see Herfindahl, 1967).

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