

EVERY 2-GROUP WITH ALL SUBGROUPS NORMAL-BY-FINITE  
IS LOCALLY FINITE

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*Abstract.* A group  $G$  has all of its subgroups normal-by-finite if  $H/H_G$  is finite for all subgroups  $H$  of  $G$ . The Tarski-groups provide examples of  $p$ -groups ( $p$  a “large” prime) of nonlocally finite groups in which every subgroup is normal-by-finite. The aim of this paper is to prove that a 2-group with every subgroup normal-by-finite is locally finite. We also prove that if  $|H/H_G| \leq 2$  for every subgroup  $H$  of  $G$ , then  $G$  contains an Abelian subgroup of index at most 8.

*Keywords:* 2-group; locally finite group; normal-by-finite subgroup; core-finite group

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## 1. INTRODUCTION

A group  $G$  is a **CF**-group (core-finite) if each of its subgroups is normal-by-finite, that is,  $H/H_G$  is finite for all  $H \leq G$ . The subgroup  $H_G$ , the core of  $H$  in  $G$ , defined by

$$H_G = \bigcap_{g \in G} H^g,$$

is the biggest normal subgroup of  $G$  contained in  $H$ .

If  $G$  is a **CF**-group, we denote by  $\sigma(G)$  the  $\sup_{H \leq G} |H : H_G|$  and we say that  $G$  is **BCF** (boundedly core-finite) if  $\sigma(G) < \infty$ .

The main Theorem in [1] states that every locally finite **CF**-group is Abelian-by-finite and **BCF**. In order to prove the previous result, the hypothesis that  $G$  is locally finite is essential, as indicated by the existence of so-called Tarski-groups, for instance the examples due to Rips and Ol’shanskii (see [5]) of infinite groups all of whose nontrivial subgroups have prime order.

The aim of this paper is to provide an elementary proof of the following results.

**Theorem A.** *Let  $G$  be a 2-group. If  $G$  is a **CF**-group, then  $G$  is locally finite.*

As a consequence of Theorem A and the main result of [1] we have that every **CF** 2-group is Abelian-by-finite and **BCF**.

In the particular case of 2-groups  $G$  with  $\sigma(G) \leq 2$  we can provide a proof that  $G$  is locally finite and hypercentral in a direct way (see Lemma 8). We also extend to all groups a result proved in [7] for finite groups.

**Theorem B.** *Let  $G$  be a group. If  $\sigma(G) \leq 2$ , then  $G$  contains an Abelian subgroup of index at most 8.*

## 2. PROOFS

A quasicyclic 2-group is a group of type  $C_{2^\infty}$ , that is, an Abelian group with generators  $x_1, x_2, \dots, x_n, \dots$  and defining relations  $x_1^2 = 1$  and  $x_{i+1}^2 = x_i$  for all  $i > 1$ . A group of type  $C_{2^\infty}$  is not finitely generated and has every proper subgroup cyclic of infinite index; in particular, if a **CF**-group  $G$  contains a quasicyclic 2-group  $C$ , then  $C \trianglelefteq G$ .

Let  $G$  be a group and let  $\Lambda$  be a well-ordered set; an indexed set  $(G_\lambda)_{\lambda \in \Lambda}$  of subgroups of  $G$  is an ascending series if whenever  $\lambda, \mu \in \Lambda$  and  $\lambda \leq \mu$ , then  $G_\lambda \leq G_\mu$ .

**Lemma 1.** *If a group  $G$  is the union of an ascending series  $(G_\lambda)_{\lambda \in \Lambda}$  of locally finite groups, then  $G$  is locally finite.*

*Proof.* Trivial (see the proof of Lemma 1.A.2 in [3]). □

**Lemma 2.** *Every periodic hyperabelian group is locally finite.*

*Proof.* It is clear that a periodic Abelian group is locally finite. Since extensions of locally finite groups by locally finite groups are locally finite (Lemma 1.A.2 of [3] or 14.3.1 of [6]), the conclusion follows by Lemma 1. □

The following result is the key to prove our Theorem A.

**Lemma 3.** *Let  $G$  be an infinite 2-group containing no subgroups of type  $C_{2^\infty}$ . Then the centralizer of every finite subgroup of  $G$  is infinite.*

*Proof.* This is Theorem C of [4]. □

**Proof of Theorem A.** Let  $G$  be an infinite 2-group which is **CF**. Since every quotient of a **CF**-group is trivially a **CF**-group, in order to prove that  $G$  is locally finite, by Lemma 2 it suffices to prove that  $G$  contains a nontrivial normal Abelian subgroup. We must distinguish between two cases.

(I)  $G$  contains a subgroup  $A$  of type  $C_{2^\infty}$ .

Since  $|A : A_G|$  is finite and  $A$  does not contain proper subgroups of finite index, we have  $A_G = A$  and hence  $A$  is a nontrivial normal Abelian subgroup of  $G$ .

(II)  $G$  contains no subgroups of type  $C_{2^\infty}$ .

By the Zorn's Lemma in  $G$  there is a maximal Abelian subgroup  $A$ . If  $A$  is finite, then by Lemma 3,  $C_G(A)$  is infinite. Let  $g \in C_G(A) \setminus A$ , then clearly  $\langle A, g \rangle$  is Abelian, against the hypothesis that  $A$  is maximal. Hence  $A$  is infinite and since  $|A : A_G| < \infty$ , we have that  $A_G$  is infinite. So  $A_G$  is a nontrivial normal Abelian subgroup of  $G$ .

This proves Theorem A. □

**Remark 4.** Let  $G$  be a finite 2-group.

- (a) If  $\sigma(G) = 1$ , then  $G$  is Abelian or Hamiltonian (for the structure of Hamiltonian groups see 5.3.7 of [6]), in particular,  $G$  contains an Abelian normal subgroup of index 2.
- (b) In [7] it is proved that if  $\sigma(G) \leq 2$ , then  $G$  contains an Abelian normal subgroup of index at most 4 and this bound is sharp.
- (c) In [2] it is proved that if  $\sigma(G) \leq 2^s$ , then  $G$  contains an Abelian subgroup of index at most  $2^t$  with

$$t \leq \frac{1}{2}(s+1)(2s^3 + 7s^2 + 9s + 2)(s^3 + 2s^2 + 3s + 3).$$

Using an inverse limit argument (see Proposition 1.K.2 of [3] and the Remark that follows) it is not difficult to check that all previous results are valid if  $G$  is locally finite, and hence, by our Theorem A, for all 2-groups.

By proving Theorem B we extend the Corollary of [7] to infinite groups. In the proof we freely make use of the fact that if  $G$  is a **BCF**-group and  $H \leq G$ , then

$$\sigma(H) \leq \sigma(G),$$

moreover if  $H \trianglelefteq G$ , then

$$\sigma(G/H) \leq \sigma(G).$$

**Lemma 5.** *Let  $G$  be a group with  $\sigma(G) \leq 2$  and let  $A$  be the subgroup of  $G$  generated by all infinite cyclic normal subgroups and by all cyclic subgroups of odd order. Then  $A$  is Abelian.*

*Proof.* Finite odd-order subgroups of  $G$  are normal and also Dedekind groups, which implies that the elements of  $G$  of odd order generate an abelian normal subgroup of  $G$ .

Let  $x \in G$  with  $o(x) = \infty$  and  $\langle x \rangle \trianglelefteq G$ . The only nonidentity element  $\alpha$  of  $\text{Aut}(\langle x \rangle)$  inverts  $x$ ; if  $a \in G$  and  $x^a = x^{-1}$ , then  $\langle x \rangle \cap \langle a \rangle \leq C_{\langle x \rangle}(\alpha) = 1$ . Thus,  $x$  is centralized by every odd order element and by every infinite cyclic normal subgroup of  $G$ . This shows that  $A$  is abelian.  $\square$

**Lemma 6.** *Let  $G$  be a nonperiodic group with  $\sigma(G) \leq 2$  and let  $A$  be the subgroup generated by all normal infinite cyclic subgroups of  $G$  and by all cyclic subgroups of odd order of  $G$ . Then for each  $g \in G$  either  $a^g = a$  for all  $a \in A$  or  $a^g = a^{-1}$  for all  $a \in A$ . Therefore  $|G : C_G(A)| \leq 2$  and moreover  $g \in C_G(A)$  for every  $g \in G$  of infinite order.*

*Proof.* By Lemma 5,  $A$  is Abelian and hence if  $g \in G$  induces by conjugation a nontrivial automorphism on  $A$ , then  $g^2 \in A$ .

Let  $x \in G$  with  $o(x) = \infty$  and let  $u \in A$  of odd order. We have  $\langle x^2 \rangle \trianglelefteq G$ , hence  $[u, x^2] = 1$  and we can write  $(x^2u)^2 = x^4u^2$ , so  $o(x^2u) = \infty$  and  $\langle (x^2u)^2 \rangle \trianglelefteq G$ . Now

$$\langle u \rangle = \langle u^2 \rangle \leq \langle x^2, (x^2u)^2 \rangle.$$

Accordingly, if  $G$  is a nonperiodic group, then the subgroup  $A$  is the one generated by the normal infinite cyclic subgroups of  $G$ . Let  $\langle x \rangle \trianglelefteq G$  and  $\langle y \rangle \trianglelefteq G$  with  $o(x) = \infty = o(y)$ . If there is  $g \in G$  such that  $x^g = x^{-1}$  and  $y^g = y$ , then  $\langle x, y \rangle = \langle x \rangle \times \langle y \rangle$ , so  $o(xy) = \infty$ , but  $\langle (xy)^2 \rangle$  is not normal in  $G$ . This shows that  $a^g = a^{-1}$  for every  $a \in A$  and hence  $|G : C_G(A)| \leq 2$ .  $\square$

**Remark 7.** From Lemmas 5 and 6 it follows that every group  $G$  with  $\sigma(G) \leq 2$  and without elements of order 2 is Abelian.

**Lemma 8.** *Let  $G$  be a 2-group. If  $\sigma(G) \leq 2$ , then  $G$  is hypercentral.*

*Proof.* Suppose  $G \neq 1$ . If  $G$  has no elements of order 4, then  $G$  is elementary Abelian and the claim is proved. If there is an element  $g \in G$  with  $o(g) = 4$ , then  $\langle g^2 \rangle$  is a normal subgroup of order 2 of  $G$  and hence  $g^2 \in Z(G)$ , so  $Z(G) \neq 1$ .  $\square$

**Lemma 9.** *Let  $G$  be a hypercentral periodic group. If  $\sigma(G) \leq 2$ , then  $G$  contains an Abelian normal subgroup of index at most 4.*

**Proof.** Since  $G$  is hypercentral, it is locally finite and, by Lemma 5, we can write  $G = R_1 \times R_2$  with  $R_1$  an Abelian 2'-group and  $R_2$  a locally finite (hypercentral) group. By Remark 4(b),  $R_2$  contains a normal Abelian subgroup  $S$  of index at most 4. We can conclude that  $R_1 \times S$  is a normal Abelian subgroup of index at most 4 in  $G$ .  $\square$

**Proof of Theorem B.** Suppose that  $G$  is a periodic group and let  $A$  be the subgroup of  $G$  generated by all elements of odd order. By Lemma 5,  $A$  is Abelian, and hence locally finite. Since  $G/A$  is a 2-group with  $\sigma(G/A) \leq 2$ , it is locally finite and hence we can deduce that  $G$  is locally finite. By Corollary of [7], every finite group  $X$  with  $\sigma(X) \leq 2$  contains an Abelian subgroup of index at most 8, applying Proposition 1.K.2 of [3] we obtain that also  $G$  has an Abelian subgroup of index at most 8. So in this case the claim is proved.

From now we suppose that  $G$  contains elements of infinite order.

Let  $A$  be the subgroup of  $G$  generated by all normal infinite cyclic subgroups of  $G$  and by all cyclic subgroups of odd order of  $G$  and let  $C = C_G(A)$  (by the proof of Lemma 6,  $A$  is actually generated by the normal infinite cyclic subgroups of  $G$ ). By construction we have that  $A$  is normal and  $G/A$  is a 2-group, by Lemma 5 we have that  $A$  is Abelian and by Lemma 6 that  $|G : C| \leq 2$ .

First of all we prove that if  $G = C$ , then  $G$  contains a normal Abelian subgroup of index at most 4. In this case  $G$  is hypercentral because  $G/A$  is a 2-group, which is hypercentral by Lemma 8, and  $A \leq Z(G)$  by hypothesis.

Let  $a \in A$  be such that  $\langle a \rangle \trianglelefteq G$  and  $o(a) = \infty$ .

Let  $h \in G$ , then  $[h, a] = 1$ , moreover if  $o(h) < \infty$ , then  $o(ah) = \infty$  and  $a^2h^2 = (ah)^2 \in A$ , so  $h^2 \in A$ . Hence for every  $g \in G$  we have  $g^2 \in A$  and  $G/A$  is an elementary Abelian 2-group. Since  $A \leq Z(G)$  for  $g, h \in G$ , we can write

$$[g, h]^2 = [g^2, h] = 1$$

and this shows  $G'$  to be an elementary Abelian 2-subgroup of  $A$ . By the Zorn's Lemma in  $A$  there is a free Abelian subgroup of maximal rank  $Q$ , moreover we can suppose  $Q \trianglelefteq G$  because since  $|Q : Q_G| \leq 2$ , we can consider  $Q_G$  instead of  $Q$ . Then  $A/Q$  is an Abelian periodic group. Let  $T$  be the full preimage in  $G$  of the 2'-Hall subgroup of  $A/Q$ , then  $T$  does not have elements of finite even order and  $G/T$  is a 2-group. Moreover,  $G' \cap T = 1$  and hence the full preimage of any Abelian subgroup of  $G/T$  is an Abelian subgroup of  $G$ . Since  $G/T$  is periodic and hypercentral, then, by Lemma 9, it contains a normal Abelian subgroup  $S/T$  of index at most 4. The full preimage  $S$  of  $S/T$  in  $G$  is a normal Abelian subgroup of index at most 4 of  $G$ .

Suppose now  $|G : C| = 2$ . For the previous argument we know that in  $C$  there is an Abelian (normal) subgroup  $S$  of index at most 4 and hence  $|G : S| \leq 8$ .  $\square$

**Remark 10.** The bound found in Theorem B is best possible, as has been shown in the proof of Corollary of [7].

**Example (i).** Let  $G = \langle x_i, y; x_i^y = x_i^{-1} \rangle$  be the semidirect product of the quasicyclic 2-group  $\langle x_1, x_2, \dots, x_n, \dots \rangle$  by the infinite cyclic group  $\langle y \rangle$ , then  $G$  is a nonperiodic **CF**-group. The element  $x_n y^2 \in G$  has infinite order and it is easy to prove that  $|\langle x_n y^2 \rangle : \langle x_n y^2 \rangle_G| = 2^{n-1}$ , so  $G$  is not a **BCF**-group. This shows that the main theorem of [1] cannot be extended to nonperiodic groups.

**Remark 11.** Let  $G$  be a group and suppose that for every  $g \in G$  we have  $|\langle g \rangle : \langle g \rangle_G| \leq 2$ . Then the subgroup  $N = \langle g^2; g \in G \rangle$  is Abelian or Hamiltonian and  $G/N$  has exponent 2, so it is elementary Abelian. Hence  $G' \leq N$  and  $|G''| \leq 2$  (in particular if  $G$  is periodic, then it is locally finite), but it can happen that  $\sigma(G) = \infty$ , as shown in Example (ii).

**Example (ii).** Let

$$G = \bigoplus_{i \in \mathbb{N}} D_i$$

with  $D_i = \langle x_i, y_i; x_i^A, y_i^2, x_i^{y_i} x_i \rangle$  isomorphic to the dihedral group of order 8. It is clear that if  $g \in G$ , then  $|\langle g \rangle : \langle g \rangle_G| \leq 2$ , but  $G$  does not possess Abelian subgroups of finite index. Moreover, if  $n \in \mathbb{N}$ , the elementary Abelian subgroup  $H = \langle y_1, y_2, \dots, y_n \rangle$  of  $G$  is such that  $|H| = 2^n$  and  $H_G = 1$ , in particular  $\sigma(G) = \infty$ .

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### References

- [1] *J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith, J. Wiegold*: Groups with all subgroups normal-by-finite. *J. Aust. Math. Soc., Ser. A* 59 (1995), 384–398. [zbl](#) [MR](#)
- [2] *G. Cutolo, E. I. Khukhro, J. C. Lennox, S. Rinauro, H. Smith, J. Wiegold*: Locally finite groups all of whose subgroups are boundedly finite over their cores. *Bull. Lond. Math. Soc.* 29 (1997), 563–570. [zbl](#) [MR](#) [doi](#)
- [3] *O. H. Kegel, B. A. F. Wehrfritz*: Locally Finite Groups. North-Holland Mathematical Library 3, North-Holland Publishing, Amsterdam, 1973. [zbl](#) [MR](#)
- [4] *J. C. Lennox, A. Mohammadi Hassanabadi, A. G. R. Stewart, J. Wiegold*: Nilpotent extensibility and centralizers in infinite 2-groups. Proceedings of the Second International Group Theory Conference (Bressanone, 1989). *Rend. Circ. Mat. Palermo (2) Suppl.* No. 23, 1990, pp. 209–219. [zbl](#) [MR](#)
- [5] *A. Yu. Ol'shanskii*: Geometry of Defining Relations in Groups. Mathematics and Its Applications. Soviet Series 70, Kluwer Academic Publishers, Dordrecht, 1991. [zbl](#) [MR](#) [doi](#)
- [6] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80, Springer, New York, 1996. [zbl](#) [MR](#) [doi](#)
- [7] *B. Wilkens*: More on core-2 2-groups. *J. Group Theory* 20 (2017), 193–225. [zbl](#) [MR](#) [doi](#)

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