# BOOLEAN-LIKE ALGEBRAS OF FINITE DIMENSION

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Abstract. We introduce Boolean-like algebras of dimension $n$ ($n$BA) having $n$ constants $\mathsf{e}_1, \ldots, \mathsf{e}_n$, and an $(n+1)$-ary operation $q$ (a "generalised if-then-else") that induces a decomposition of the algebra into $n$ factors through the so-called $n$-central elements. Varieties of $n$BAs share many remarkable properties with the variety of Boolean algebras and with primal varieties. Exploiting the concept of central element, we extend the notion of Boolean power to that of semiring power and we prove two representation theorems: (i) Any pure $n$BA is isomorphic to the algebra of $n$-central elements of a Boolean vector space; (ii) Any member of a variety of $n$BAs with one generator is isomorphic to a Boolean power of this generator. This gives a new proof of Foster's theorem on primal varieties. The $n$BAs provide the algebraic framework for generalising the classical propositional calculus to the case of $n$ - perfectly symmetric - truth-values. Every finite-valued tabular logic can be embedded into such an $n$-valued propositional logic, $n$CL, and this embedding preserves validity. We define a confluent and terminating first-order rewriting system for deciding validity in $n$CL, and, via the embeddings, in all the finite tabular logics.

## 1. Introduction

We aim at bridging several different areas of logic, algebra and computation — the algebraic analysis of conditional statements in programming languages, the theory of factorisations of algebras, the theory of Boolean vector spaces, the investigation into generalisations of classical logic — most of which somehow revolve around the main concept that lies at the crossroads of the three disciplines: the notion of *Boolean algebra*.

There is a thriving literature on abstract treatments of the *if-then-else* construct of computer science, starting with McCarthy's seminal investigations [18]. On the algebraic side, one of the most influential approaches originated with the work of Bergman [2]. Bergman modelled the if-then-else by considering Boolean algebras acting on sets: if the Boolean algebra of actions is the 2-element algebra, one simply puts $1(a,b) = a$ and $0(a,b) = b$. In [20], on the other hand, some of the present authors took their cue from Dicker's axiomatization of Boolean algebras in the language with the if-then-else as primitive [6]. Accordingly, this construct was treated as a proper algebraic operation $q^{\mathbf{A}}$ on algebras $\mathbf{A}$ whose type contains, besides the ternary term $q$, two constants 0 and 1, and having the property that for every $a, b \in A$, $q^{\mathbf{A}}(1^{\mathbf{A}}, a, b) = a$ and $q^{\mathbf{A}}(0^{\mathbf{A}}, a, b) = b$. Such algebras, called *Church algebras* in [20], will be termed here *algebras of dimension* 2.

The reason for this denomination is as follows. At the root of the most important results in the theory of Boolean algebras (including Stone's representation theorem) there is the simple observation that every element $e \neq 0, 1$ of a Boolean algebra $\mathbf{B}$ decomposes $\mathbf{B}$ as a Cartesian product $[0, e] \times [e, 1]$ of *two* nontrivial Boolean algebras. In the more general context of algebras

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of dimension 2, we say that an element $e$ of an algebra $\mathbf{A}$ of dimension 2 is called 2-*central* if $\mathbf{A}$ can be decomposed as the product $\mathbf{A}/\theta(e,0) \times \mathbf{A}/\theta(e,1)$, where $\theta(e,0)$ ($\theta(e,1)$) is the smallest congruence on $\mathbf{A}$ that collapses $e$ and 0 ($e$ and 1). Algebras of dimension 2 where every element is 2-central were called *Boolean-like algebras* in [20], since the variety of all such algebras in the language $(q, 0, 1)$ is term-equivalent to the variety of Boolean algebras. In this paper, on the other hand, they will be called *Boolean-like algebras of dimension* 2. Against this backdrop, it is tempting to generalise the previous approach to algebras $\mathbf{A}$ having $n$ designated elements $\mathsf{e}_1, \dots, \mathsf{e}_n$ ($n \geq 2$) and an $n+1$-ary operation $q$ (a sort of "generalised if-then-else") that induces a decomposition of $\mathbf{A}$ into $n$, rather than just 2, factors. These algebras will be called, naturally enough, *algebras of dimension* $n$ ($n$DA), while algebras $\mathbf{A}$ all of whose elements induce an $n$-ary factorisation of this sort are given the name of *Boolean-like algebras of dimension* $n$ ($n$BAs). Free $\mathcal{V}$-algebras (for $\mathcal{V}$ a variety), lambda algebras, semimodules over semirings - hence, in particular, Boolean vector spaces - give rise to algebras which, in general, have dimension greater than 2.

Apart from Boolean algebras, many algebras investigated in classical mathematics, like rings with unit or ortholattices, have dimension 2. We succeeded in generalising Boolean algebras to $n$-dimensional Boolean-like algebras. It would be worthwhile to explore whether the latter classes of algebras also admit of meaningful $n$-dimensional counterparts, a goal we reserve for future investigation.

Varieties of $n$BAs share many remarkable properties with the variety of Boolean algebras. In particular, we show that any variety of $n$BAs is generated by the $n$BAs of finite cardinality $n$. In the pure case (i.e., when the type includes just the generalised if-then-else $q$ and the $n$ constants), the variety is generated by a unique algebra $\mathbf{n}$ of universe $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$, so that any pure $n$BA is, up to isomorphism, a subalgebra of $\mathbf{n}^I$, for a suitable set $I$. Another remarkable property of the 2-element Boolean algebra is the definability of all finite Boolean functions in terms e.g. of the connectives AND, OR, NOT. This property is inherited by the algebra $\mathbf{n}$: all finite functions on $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$ are term-definable, so that the variety of pure $n$BAs is primal. More generally, a variety of an arbitrary type with one generator is primal if and only if it is a variety of $n$BAs.

Next, we parlay the theory of $n$-central elements into an extension to arbitrary semirings of the technique of Boolean powers. We algebraically define the semiring power $\mathbf{E}[R]$ of an arbitrary algebra $\mathbf{E}$ by a semiring $R$ as an algebra over the set of central elements of a certain semimodule. We obtain the following results: (i) If the semiring $R$ is a Boolean algebra, then the algebraically defined semiring power is isomorphic to the classical, topologically defined, Boolean power [4]; (ii) For every semiring $R$, the semiring power $\mathbf{E}[R]$ is isomorphic to the Boolean power of $\mathbf{E}$ by the Boolean algebra of the complemented and commuting elements of $R$.

We conclude the algebraic part of the article with two representation theorems. We first show that any pure $n$BA $\mathbf{A}$ contains a Boolean algebra $B_{\mathbf{A}}$; then we represent $\mathbf{A}$, up to isomorphism, as the $n$BA of $n$-central elements of the Boolean vector space $(B_{\mathbf{A}})^n$. In the second representation theorem it is shown that any $n$BA in a variety of $n$BAs with one generator is isomorphic to a Boolean power. Foster's theorem on primal varieties [4, Thm. 7.4], according to which any member of a variety generated by a primal algebra is a Boolean power of the generator, follows as a corollary.

Just like Boolean algebras are the algebraic counterpart of classical logic CL, for every $n \geq 2$ we define a logic $n$CL whose algebraic counterpart are $n$BAs. We show the complete symmetry of the truth values $\mathsf{e}_1, \dots, \mathsf{e}_n$, supporting the idea that $n$CL is the right generalisation of classical logic from dimension 2 to dimension $n$. Then the universality of $n$CL is obtained by conservatively translating any $n$-valued tabular logic with a single designated value into it.

We define a terminating and confluent term rewriting system to test the validity of formulas of $n$CL by rewriting. By the universality of $n$CL, in order to check whether a formula $\phi$ is valid in a $n$-valued tabular logic, it is enough to see whether the normal form $\phi^*$ of the translation is the designated value of $n$CL. Our rewriting rules are very tightly related to the equivalence transformation rules of multi-valued decision diagrams [11, 19]. Our approach generalises Zantema and van de Pol's work [28] on rewriting of binary decision diagrams and the work by Salibra et al. [21] on rewriting terms of factor varieties with decomposition operators. We point out that in [21] tabular logics were translated in terms of decomposition operators, very much in the spirit of what we are doing here. Nevertheless, due to the lack of an underlying logic and to the absence of the $q$ operator, those translations were ad hoc.

## 2. Preliminaries

The notation and terminology in this paper are pretty standard. For concepts, notations and results not covered hereafter, the reader is referred to [4, 17] for universal algebra and to [8] for semirings and semimodules. As to the rest, superscripts that mark the difference between operations and operation symbols will be dropped whenever the context is sufficient for a disambiguation.

If $\tau$ is an algebraic type, an algebra $\mathbf{A}$ of type $\tau$ is called *a $\tau$-algebra*, or simply an algebra when $\tau$ is clear from the context. An algebra is *trivial* if its carrier set is a singleton set.

$\mathrm{Con}(\mathbf{A})$ is the lattice of all congruences on $\mathbf{A}$, whose bottom and top elements are, respectively, $\Delta = \{(x,x) : x \in A\}$ and $\nabla = A \times A$. The same symbol will also denote the universe of the same lattice. Given $a, b \in A$, we write $\theta(a,b)$ for the smallest congruence $\theta$ such that $(a,b) \in \theta$.

We say that an algebra $\mathbf{A}$ is: (i) *subdirectly irreducible* if the lattice $\mathrm{Con}(\mathbf{A})$ has a unique atom; (ii) *simple* if $\mathrm{Con}(\mathbf{A}) = \{\Delta, \nabla\}$; (iii) *directly indecomposable* if $\mathbf{A}$ is not isomorphic to a direct product of two nontrivial algebras. Any simple algebra is subdirectly irreducible and any subdirectly irreducible algebra is directly indecomposable.

A class $\mathcal{V}$ of $\tau$-algebras is a *variety* if it is closed under subalgebras, direct products and homomorphic images. By Birkhoff's theorem [4, Thm. 2.11.9] a class of algebras is a variety if and only if it is an equational class. If $K$ is a class of $\tau$-algebras, the variety $V(K)$ generated by $K$ is the smallest variety including $K$. If $K = \{\mathbf{A}\}$ we write $V(\mathbf{A})$ for $V(\{\mathbf{A}\})$.

### 2.1. Factor Congruences and Decomposition.

**Definition 1.** *A tuple $(\theta_1, \dots, \theta_n)$ of congruences on $\mathbf{A}$ is a family of* complementary factor congruences *if the function $f : \mathbf{A} \to \prod_{i=1}^{n} \mathbf{A}/\theta_i$ defined by $f(a) = (a/\theta_1, \dots, a/\theta_n)$ is an isomorphism. When $|I| = 2$, we say that $(\theta_1, \theta_2)$ is a pair of complementary factor congruences.*

A *factor congruence* is any congruence which belongs to a family of complementary factor congruences.

**Theorem 1.** *A tuple $(\theta_1, \dots, \theta_n)$ of congruences on $\mathbf{A}$ is a family of complementary factor congruences exactly when:*

(1) $\bigcap_{1 \le i \le n} \theta_i = \Delta$;
(2) $\forall (a_1, \dots, a_n) \in A^n$, *there is $u \in A$ such that $a_i \theta_i u$, for all $1 \le i \le n$.*

Therefore $(\theta_1, \theta_2)$ is a pair of complementary factor congruences if and only if $\theta_1 \cap \theta_2 = \Delta$ and $\theta_1 \circ \theta_2 = \nabla$. The pair $(\Delta, \nabla)$ corresponds to the product $\mathbf{A} \cong \mathbf{A} \times \mathbf{1}$, where $\mathbf{1}$ is a trivial algebra; obviously $\mathbf{1} \cong \mathbf{A}/\nabla$ and $\mathbf{A} \cong \mathbf{A}/\Delta$. The set of factor congruences of $\mathbf{A}$ is not, in general, a sublattice of $\mathrm{Con}(\mathbf{A})$.

Factor congruences can be characterised in terms of certain algebra homomorphisms called *decomposition operators* (see [17, Def. 4.32] for additional details).

**Definition 2.** *A* decomposition operator *on an algebra* $\mathbf{A}$ *is a function* $f : A^n \to A$ *satisfying the following conditions:*

- **D1:** $f(x, x, \dots, x) = x$;
- **D2:** $f(f(x_{11}, x_{12}, \dots, x_{1n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})) = f(x_{11}, \dots, x_{nn})$;
- **D3:** *$f$ is an algebra homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$.*

Axioms (D1)-(D3) can be equationally expressed.

There is a bijective correspondence between families of complementary factor congruences and decomposition operators, and thus, between decomposition operators and factorisations of an algebra.

**Theorem 2.** *Any decomposition operator $f : \mathbf{A}^n \to \mathbf{A}$ on an algebra $\mathbf{A}$ induces a family of complementary factor congruences $\theta_1, \dots, \theta_n$, where each $\theta_i \subseteq A \times A$ is defined by:*

$$a\ \theta_i\ b \quad \textit{iff} \quad f(a, \dots, a, b, a, \dots, a) = a \qquad (b \textit{ at position } i).$$

*Conversely, any family $\theta_1, \dots, \theta_n$ of complementary factor congruences induces a decomposition operator $f$ on $\mathbf{A}$: $f(a_1, \dots, a_n) = u$ iff $a_i\, \theta_i\, u$, for all $i$, where such an element $u$ is provably unique.*

2.2. **Semimodules and Boolean vector spaces.** A *semiring* $R$ [8] is an algebra $(R, +, \cdot, 0, 1)$ such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, and the following equations hold:

- **SR1:** $x0 = 0x = 0$;
- **SR2:** $x(y + z) = xy + yz$;
- **SR3:** $(y + z)x = yx + zx$.

Thus, in particular, rings with unit are semirings in which every element has an additive inverse. Also, any bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ is a semiring.

An $n$-tuple of elements $r_1, \dots, r_n$ of a semiring is *fully orthogonal* if $r_1 + \dots + r_n = 1$ and $r_i r_j = 0$ for every $i \neq j$.

**Definition 3.** *If $R$ is a semiring, a* (left) $R$-semimodule *(see [8]) is a commutative monoid $(V, +, \mathbf{0})$ for which we have a function $R \times V \to V$, denoted by $(r, \mathbf{v}) \mapsto r\mathbf{v}$ and called* scalar multiplication*, which satisfies the following conditions, for all elements $r, s \in R$ and all elements $\mathbf{v}, \mathbf{w} \in V$:*

- **SM1:** $(r \cdot s)\mathbf{v} = r(s\mathbf{v})$;
- **SM2:** $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$ *and* $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$;
- **SM3:** $0\mathbf{v} = \mathbf{0}$ *and* $1\mathbf{v} = \mathbf{v}$.

An $R$-semimodule is called: (i) a *module* if $R$ is a ring; (ii) a *vector space* if $R$ is a field; (iii) a *Boolean vector space* (see [9] for basic facts on Boolean vector spaces) if $R$ is a Boolean algebra. The elements of an $R$-semimodule are called *vectors*.

If $V$ is an $R$-semimodule and $E \subseteq V$ then we denote by $R\langle E\rangle = \{\sum_{i=1}^n r_i \mathsf{e}_i : r_i \in R, \mathsf{e}_i \in E, n \in \mathbb{N}\}$, the set of linear combinations of elements of $E$. The set $E$ is called a *free basis* of $V$ if $R\langle E\rangle = V$ and each vector in $V$ can be expressed as a linear combination of elements in $E$ in exactly one way. An $R$-semimodule having a free basis $E$ is denoted by $R\langle E\rangle$ and is called the *$R$-semimodule freely generated by $E$*.

If $\mathbf{v} \in R\langle E\rangle$, then there exist $\mathsf{e}_1, \dots, \mathsf{e}_n \in E$ and scalars $r_1, \dots, r_n \in R$ such $\mathbf{v} = \sum_{i=1}^n r_i \mathsf{e}_i$.

The *coordinates* $v_{\mathsf{d}}$ ($\mathsf{d} \in E$) of a vector $\mathbf{v} = \sum_{i=1}^{n} r_i \mathsf{e}_i$ w.r.t. the free basis $E$ are defined as follows:

$$v_{\mathsf{d}} = \begin{cases} r_i & \text{if } \mathsf{d} = \mathsf{e}_i \ (i = 1, \dots, n) \\ 0 & \text{if } \mathsf{d} \neq \mathsf{e}_1, \dots, \mathsf{e}_n. \end{cases}$$

Each element $\mathbf{v}$ of $R\langle E\rangle$ can be represented by the formal series $\mathbf{v} = \sum_{\mathsf{e}\in E} v_{\mathsf{e}}\mathsf{e}$, where almost all scalars $v_{\mathsf{e}}$ coincide with 0.

2.3. **Tabular Logics.** In logic the operation symbols of an algebraic type $\tau$ are considered as logical connectives. Then the term algebra $\mathbf{T}_\tau(X)$ over the countable set $X$ of generators is also called the algebra of $\tau$-formulas.

**Definition 4.** *A finitary* logic *of type* $\tau$ [3] *is a substitution-invariant consequence relation* $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(\mathbf{T}_\tau(X)) \times \mathbf{T}_\tau(X)$ *such that:*

- **1.:** $\alpha \vdash_{\mathrm{L}} \alpha$ *(reflexivity);*
- **2.:** *If* $\Gamma \vdash_{\mathrm{L}} \alpha$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash_{\mathrm{L}} \alpha$ *(monotonicity);*
- **3.:** *If* $\Gamma \vdash_{\mathrm{L}} \alpha$ *and* $\Delta \vdash_{\mathrm{L}} \gamma$ *for every* $\gamma \in \Gamma$*, then* $\Delta \vdash_{\mathrm{L}} \alpha$ *(cut);*
- **4.:** *If* $\Gamma \vdash_{\mathrm{L}} \alpha$ *and* $\sigma$ *is an endomorphism of* $\mathbf{T}_\tau(X)$*, then* $\sigma(\Gamma) \vdash_{\mathrm{L}} \sigma(\alpha)$ *(substitution-invariance);*
- **5.:** *If* $\Gamma \vdash_{\mathrm{L}} \alpha$*, then there exists a finite* $\Delta \subseteq \Gamma$ *such that* $\Delta \vdash_{\mathrm{L}} \alpha$ *(finitarity).*

**Definition 5.** *If* $\tau$ *is a type, a* $\tau$-logical matrix [3] *(or simply a* logical matrix*, when* $\tau$ *is understood) is a pair* $(\mathbf{A}, F)$*, where* $\mathbf{A}$ *is a* $\tau$*-algebra and* $F \subseteq A$*. The elements in* $F$ *are said to be* designated. *If* $F = \{\mathsf{t}\}$ *is a singleton, we write* $(\mathbf{A}, \mathsf{t})$ *for* $(\mathbf{A}, \{\mathsf{t}\})$.

For any $\tau$-logical matrix $(\mathbf{A}, F)$ and set $\Gamma \cup \{\phi\}$ of formulas, we write $\Gamma \models_{(\mathbf{A},F)} \phi$ in case for any homomorphism $h : \mathbf{T}_\tau(X) \to \mathbf{A}$, if $h(\psi) \in F$ for any $\psi \in \Gamma$, then $h(\phi) \in F$. If $M$ is a class of logical matrices, we write $\Gamma \models_M \phi$ if $\Gamma \models_{(\mathbf{A},F)} \phi$, for every $(\mathbf{A}, F) \in M$.

Logics of the form $\models_{(\mathbf{A},F)}$, where $\mathbf{A}$ is a finite $\tau$-algebra, are called *tabular*. Many well-known logics in the literature are tabular logics with the additional property that $F = \{\mathsf{t}\}$ is a singleton; in such case, $\mathsf{t}$ can be read as representing the value "true". Some examples of tabular logics are given below.

**Example 1.** 1. *Classical Logic* CL. *The logical matrix is* $(\mathbf{2}, \mathsf{t})$ *where* $\mathbf{2}$ *is the two-element Boolean algebra of truth values with universe* $\{\mathsf{f}, \mathsf{t}\}$ *and* $\mathsf{t}$ *is the designated element.*

2. *The* $n$*-valued logics under consideration (Łukasiewicz, Gödel and Post Logics) have a totally ordered set* $0 < \frac{1}{n-1} < \frac{2}{n-1} < \dots < \frac{n-2}{n-1} < 1$ *of truth values,* 1 *as designated element, and join and meet are defined by* $a \vee b = \max\{a, b\}$ *and* $a \wedge b = \min\{a, b\}$*. These logics only differ for the definition of negation and implication, which is not present in Post Logic.*

- *Łukasiewicz Logic* $\mathrm{Ł}_n$*:* $\neg a = 1 - a$; $\qquad a \to b = \min(1, 1 - a + b)$
- *Gödel Logic* $\mathrm{G}_n$*:* $a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \qquad \neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$
- *Post Logic* $\mathrm{P}_n$*:* $\neg a = \begin{cases} a - \frac{1}{n-1} & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$

**Definition 6.** *If* $\mathrm{L}, \mathrm{L}'$ *are logics of respective types* $\tau$ *and* $\tau'$*, we say that* $\mathrm{L}$ *is a conservative* expansion *of* $\mathrm{L}'$ *if the following conditions hold:*

- *Every* $k$*-ary operation symbol* $f$ *of type* $\tau'$ *is translated into a* $k$*-ary* $\tau$*-term* $f^\circ(x_1, \dots, x_k)$*;*
- *Every* $\tau'$*-term* $\phi$ *is translated into a* $\tau$*-term* $\phi^*$*:*

$$x^* = x; \qquad f(\phi_1, \dots, \phi_k)^* = f^\circ(\phi_1^*/x_1, \dots, \phi_k^*/x_k);$$

- *For all $\tau'$-terms $\Gamma \cup \{\phi\}$, $\Gamma \vdash_{\mathrm{L}'} \phi$ iff $\Gamma^* \vdash_{\mathrm{L}} \phi^*$.*

## 3. Algebras of finite dimension

In this section we introduce algebras having $n$ designated elements $\mathsf{e}_1, \dots, \mathsf{e}_n$ ($n \geq 2$) and an operation $q$ of arity $n+1$ (a sort of "generalised if-then-else") satisfying $q(\mathsf{e}_i, x_1, \dots, x_n) = x_i$. The operator $q$ induces, through the so-called $n$-central elements, a decomposition of the algebra into $n$ factors. These algebras will be called *algebras of dimension $n$*.

**Definition 7.** *An algebra $\mathbf{A}$ of type $\tau$ is an* algebra of dimension $n$ *(an $n$*DA*, for short) if there are term definable elements $\mathsf{e}_1^{\mathbf{A}}, \mathsf{e}_2^{\mathbf{A}}, \dots, \mathsf{e}_n^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ of arity $n+1$ such that, for all $b_1, \dots, b_n \in A$ and $1 \leq i \leq n$, $q^{\mathbf{A}}(\mathsf{e}_i^{\mathbf{A}}, b_1, \dots, b_n) = b_i$. A variety $\mathcal{V}$ of type $\tau$ is a* variety of algebras of dimension $n$ *if every member of $\mathcal{V}$ is an $n$*DA *with respect to the same terms $q, \mathsf{e}_1, \dots, \mathsf{e}_n$.*

If $\mathbf{A}$ is an $n$DA, then $\mathbf{A}_0 = (A, q^{\mathbf{A}}, \mathsf{e}_1^{\mathbf{A}}, \dots, \mathsf{e}_n^{\mathbf{A}})$ is the *pure reduct* of $\mathbf{A}$.

Algebras of dimension 2 were introduced as Church algebras in [15] and studied in [20]. Examples of algebras of dimension 2 are Boolean algebras (with $q(x, y, z) = (x \wedge z) \vee (\neg x \wedge y)$) or rings with unit (with $q(x, y, z) = xz + y - xy$). Next, we list some relevant examples of algebras having dimension greater than 2.

**Example 2.** *(*Free algebras) *Let $\tau$ be a type and $X = \{x_1, \dots, x_n\}$ be a finite set of variables. For all $\phi, \psi_1, \dots, \psi_n$ in the term algebra $\mathbf{T}_\tau(X)$, we define:*

$$q(\phi, \psi_1, \dots, \psi_n) = \phi\,[\psi_1/x_1, \dots, \psi_n/x_n]\,,$$

*where $\phi\,[\psi_1/x_1, \dots, \psi_n/x_n]$ denotes the term obtained from $\phi$ by replacing each occurrence of $x_i$ by $\psi_i$, for all $i$. If $\mathcal{V}$ is any variety of algebras of type $\tau$ and $\mathbf{T}_{\mathcal{V}}(X)$ is the free $\mathcal{V}$-algebra over $X$, this operation is well-defined on equivalence classes of terms in $\mathbf{T}_{\mathcal{V}}(X)$ and turns $\mathbf{T}_{\mathcal{V}}(X)$ into an $n$*DA*, with respect to $q$ and $x_1, \dots, x_n$.*

$R$-semimodules are an important source of algebras of finite dimension, as the next example shows.

**Example 3.** *(*Semimodules*) Let $V$ be an $R$-semimodule freely generated by a finite set $E = \{\mathsf{e}_1, \dots, \mathsf{e}_n\}$. Then we define an operation $q$ of arity $n+1$ as follows (for all $\mathbf{v} = \sum_{j=1}^n v_j \mathsf{e}_j$ and $\mathbf{w}^i = \sum_{j=1}^n w_j^i \mathsf{e}_j$):*

$$q(\mathbf{v}, \mathbf{w}^1, \dots, \mathbf{w}^n) = \sum_{i=1}^n v_i \mathbf{w}^i = \sum_{k=1}^n (\sum_{i=1}^n v_i \cdot w_k^i) \mathsf{e}_k.$$

*Under this definition, $V$ becomes an $n$*DA*. Each $\mathbf{w}^i$ in the definition of $q$ can be viewed as the $i$-th column vector of an $n \times n$ matrix $M$. The operation $q$ does nothing but express in an algebraic guise the application of the linear transformation encoded by $M$ to the vector $\mathbf{v}$.*

A special instance of the previous example, described below, will play a role in what follows.

**Example 4.** *(*$n$-Sets*) Let $X$ be a set. An* $n$-subset *of $X$ is a sequence $(Y_1, \dots, Y_n)$ of subsets $Y_i$ of $X$. We denote by* $\mathrm{Set}_n(X)$ *the family of all $n$-subsets of $X$.* $\mathrm{Set}_n(X)$ *can be viewed as the universe of a Boolean vector space over the powerset $\mathcal{P}(X)$ with respect to the following operations:*

$$(Y_1, \dots, Y_n) + (Z_1, \dots, Z_n) = (Y_1 \cup Z_1, \dots, Y_n \cup Z_n)$$

*and, for every $Z \subseteq X$,*

$$Z(Y_1, \dots, Y_n) = (Z \cap Y_1, \dots, Z \cap Y_n).$$

$\mathrm{Set}_n(X)$ *is freely generated by the n-sets* $\mathsf{e}_1 = (X, \emptyset, \ldots, \emptyset), \ldots, \mathsf{e}_n = (\emptyset, \ldots, \emptyset, X)$. *Thus, an arbitrary n-set* $(Y_1, \ldots, Y_n)$ *has the canonical representation* $Y_1\mathsf{e}_1 + \cdots + Y_n\mathsf{e}_n$ *as a vector. An explicit description of the q operator defined for generic semimodules in Example 3 is as follows, for all* $\mathbf{y}^i = Y_1^i\mathsf{e}_1 + \cdots + Y_n^i\mathsf{e}_n$*:*

$$q(\mathbf{y}^0, \mathbf{y}^1, \ldots, \mathbf{y}^n) = (\bigcup_{i=1}^{n} Y_i^0 \cap Y_1^i, \ldots, \bigcup_{i=1}^{n} Y_i^0 \cap Y_n^i).$$

Congruences are notoriously irksome objects in algebra. Therefore, considerable advantage is gained whenever one can find more manageable concepts that can act in their stead — like normal subgroups in the variety of groups, or, more generally, ideals in any ideal-determined variety [10]. In [27], Vaggione introduced the notion of *central element* to study algebras whose complementary factor congruences can be replaced by certain elements of their universes. If a neat description of such elements is available, one usually gets important insights into the structure theories of the algebras at issue. To list a few examples, central elements coincide with central idempotents in rings with unit, with complemented elements in $FL_{ew}$-algebras, which form the equivalent algebraic semantics of the full Lambek calculus with exchange and weakening, and with members of the centre in ortholattices. In [20], T. Kowalski and three of the present authors investigated central elements in algebras of dimension 2. Here, we generalise the idea to algebras of arbitrary finite dimension.

**Definition 8.** *If* $\mathbf{A}$ *is an* $n$DA*, then* $c \in A$ *is called* $n$-central *if the sequence of congruences* $(\theta(c, \mathsf{e}_1), \ldots, \theta(c, \mathsf{e}_n))$ *is an n-tuple of complementary factor congruences of* $\mathbf{A}$. *A central element c is* nontrivial *if* $c \notin \{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$.

By the results of Section 2, every $n$-central element $c \in A$ induces a decomposition of $\mathbf{A}$ as a direct product of the algebras $\mathbf{A}/\theta(c, \mathsf{e}_i)$, for $i \leq n$. We omit the proof of the next characterisation of $n$-central elements, which follows closely the proof for the 2-dimensional case given in [20, Prop. 3.6].

**Theorem 3.** *If* $\mathbf{A}$ *is an* $n$DA *of type* $\tau$ *and* $c \in A$*, then the following conditions are equivalent:*

1. *c is n-central;*
2. $\bigcap_{i \leq n} \theta(c, \mathsf{e}_i) = \Delta$*;*
3. *for all* $a_1, \ldots, a_n \in A$, $q(c, a_1, \ldots, a_n)$ *is the unique element such that* $a_i\ \theta(c, \mathsf{e}_i)\ q(c, a_1, \ldots, a_n)$, *for all* $1 \leq i \leq n$*;*
4. *The following conditions are satisfied:*
   **B1:** $q(c, \mathsf{e}_1, \ldots, \mathsf{e}_n) = c$.
   **B2:** $q(c, x, x, \ldots, x) = x$ *for every* $x \in A$.
   **B3:** *If* $\sigma \in \tau$ *has arity k and* $\mathbf{x}$ *is a* $n \times k$ *matrix of elements of A, then* $q(c, \sigma(\mathbf{x}_1), \ldots, \sigma(\mathbf{x}_n)) = \sigma(q(c, \mathbf{x}^1), \ldots, q(c, \mathbf{x}^k))$.
5. *The function* $f_c$*, defined by* $f_c(x_1, \ldots, x_n) = q(c, x_1, \ldots, x_n)$*, is a decomposition operator on* $\mathbf{A}$ *such that* $f_c(\mathsf{e}_1, \ldots, \mathsf{e}_n) = c$.

For any $n$-central element $c$ and any $n \times n$ matrix $\mathbf{x}$ of elements of $A$, a direct consequence of (B1)-(B3) gives

**B4:** $q(c, q(c, \mathbf{x}_1), \ldots, q(c, \mathbf{x}_n)) = q(c, x_1^1, x_2^2, \ldots, x_n^n)$.

The following proposition characterises the algebraic structure of $n$-central elements.

**Proposition 1.** *Let* $\mathbf{A}$ *be an* $n$DA. *Then the set of all n-central elements of* $\mathbf{A}$ *is a subalgebra of the pure reduct of* $\mathbf{A}$.

*Proof.* Let $a, c_1, \ldots c_n$ be $n$-central elements. It is sufficient to prove that $q(a, c_1, \ldots c_n)$ is also an $n$-central element, i.e., it satisfies axioms (B1)-(B3). □

Hereafter, we denote by $\mathbf{Ce}_n(\mathbf{A})$ the algebra $(\mathrm{Ce}_n(\mathbf{A}), q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ of all $n$-central elements of an $n$DA $\mathbf{A}$.

**Example 5.** *Let $\mathbf{A}$ be an algebra (not necessarily an nBA) of type $\tau$ and $F$ be the set of all functions from $A^n$ into $A$. Consider the nDA $\mathbf{F} = (F, \sigma^{\mathbf{F}}, q^{\mathbf{F}}, \mathsf{e}_1^{\mathbf{F}}, \dots, \mathsf{e}_n^{\mathbf{F}})_{\sigma\in\tau}$, whose operations are defined as follows (for all $f_i, g_j \in F$ and all $a_1, \dots, a_n \in A$):*

(1) $\mathsf{e}_i^{\mathbf{F}}(a_1, \dots, a_n) = a_i$;
(2) $q^{\mathbf{F}}(f, g_1 \dots, g_n)(a_1, \dots, a_n) = f(g_1(a_1, \dots, a_n) \dots, g_n(a_1, \dots, a_n))$;
(3) $\sigma^{\mathbf{F}}(f_1, \dots, f_k)(a_1, \dots, a_n) = \sigma^{\mathbf{A}}(f_1(a_1, \dots, a_n), \dots, f_k(a_1, \dots, a_n))$, *for every $\sigma \in \tau$ of arity $k$.*

*Let $\mathbf{G}$ be any subalgebra of $\mathbf{F}$ containing all constant functions. It is possible to prove that a function $f : A^n \to A$ is an $n$-central element of $\mathbf{G}$ if and only if it is an $n$-ary decomposition operator of the algebra $\mathbf{A}$ commuting with every element $g \in G$ (for every $a_{ij} \in A$):*

$$f(g(a_{11}, \dots, a_{1n}), \dots, g(a_{n1}, \dots, a_{nn})) = g(f(a_{11}, \dots, a_{n1}), \dots, f(a_{1n}, \dots, a_{nn})).$$

*The reader may consult* [22] *for the case $n = 2$.*

The following example provides an application of the $n$-central elements to lambda calculus.

**Example 6.** *(*Lambda Calculus*) We refer to Barendregt's book* [1] *for basic definitions on lambda calculus. Let* $\mathrm{Var} = \{a, b, c, a_1, \dots\}$ *be the set of variables of the lambda calculus ($\lambda$-variables, for short), and let $x, y, x_1, y_1, \dots$ be the algebraic variables ("holes" in the terminology of Barendregt's book). Once a finite set $I = \{a_1, \dots, a_n\}$ of $\lambda$-variables has been fixed, we define $n$ constants $\mathsf{e}_i$ and an operator $q_I$ as follows:*

$$\mathsf{e}_i = \lambda a_1 \dots a_n . a_i; \qquad q_I(x, y_1, \dots, y_n) = (\dots((xy_1)y_2)\dots)y_n;$$

*The term algebra of a $\lambda$-theory is an* $n$DA *with respect to the term operation $q_I$ and the constants $\mathsf{e}_1, \dots, \mathsf{e}_n$. Let $\Lambda_\beta$ be the term algebra of the minimal $\lambda$-theory $\lambda\beta$, whose lattice of congruences is isomorphic to the lattice of $\lambda$-theories. Let $\Omega = (\lambda a.aa)(\lambda a.aa)$ be the canonical looping $\lambda$-term. It turns out that $\Omega$ can be consistently equated to any other closed $\lambda$-term (see* [1, Proposition 15.3.9]*). Consider the $\lambda$-theory $T_i = \theta(\Omega, \mathsf{e}_i)$ generated by equating $\Omega$ to the constant $\mathsf{e}_i$ above defined. By Theorem 3 $\Omega$ is a nontrivial central element in the term algebra of the $\lambda$-theory $T = \bigcap_{i\le n} \theta(\Omega, \mathsf{e}_i)$, i.e., the quotient $\Lambda_\beta/T$. It is possible to prove that any model of $T$, not just its term model, is decomposable. On the other hand, the set-theoretical models of $\lambda$-calculus defined after Scott's seminal work* [24] *are indecomposable algebras. Hence, none of these set-theoretical models has $T$ as equational theory. This general incompleteness result has been proved in* [14] *for the case $n = 2$ (see also* [16]*).*

## 4. Boolean-like algebras of finite dimension

Boolean algebras are algebras of dimension 2 all of whose elements are 2-central. It turns out that, among the $n$-dimensional algebras, those algebras all of whose elements are $n$-central inherit many of the remarkable properties that distinguish Boolean algebras. We now zoom in on such algebras, which will take centre stage in this section.

**Definition 9.** *An* $n$DA $\mathbf{A}$ *is called a* Boolean-like algebra of dimension $n$ *(*$n$BA*, for short) if every element of $A$ is $n$-central.*

By Proposition 1, the algebra $\mathbf{Ce}_n(\mathbf{A})$ of all $n$-central elements of an $n$DA $\mathbf{A}$ is a canonical example of $n$BA.

The class of all $n$BAs of type $\tau$ is a variety of $n$DAs axiomatised by the identities B1-B3 in Theorem 3.

Boolean-like algebras of dimension 2 were introduced in [20] with the name "Boolean-like algebras". *Inter alia*, it was shown in that paper that the variety of pure Boolean-like algebras of dimension 2 is term-equivalent to the variety of Boolean algebras.

**Example 7.** *The algebra* $\mathbf{n} = (\{\mathsf{e}_1, \dots, \mathsf{e}_n\}, q^{\mathbf{n}}, \mathsf{e}_1^{\mathbf{n}}, \dots, \mathsf{e}_n^{\mathbf{n}})$*, where*

$$q^{\mathbf{n}}(\mathsf{e}_i, x_1, \dots, x_n) = x_i$$

*for every* $i \leq n$*, is a pure* $n$BA*.*

**Example 8.** *(*$n$-Partitions*) Let* $X$ *be a set. An* $n$-partition *of* $X$ *is an* $n$*-subset* $(Y_1, \dots, Y_n)$ *of* $X$ *such that* $\bigcup_{i=1}^n Y_i = X$ *and* $Y_i \cap Y_j = \emptyset$ *for all* $i \neq j$*. The set of* $n$*-partitions of* $X$ *is closed under the* $q$*-operator defined in Example 4 and constitutes the algebra of all* $n$*-central elements of the Boolean vector space* $\mathrm{Set}_n(X)$ *of all* $n$*-subsets of* $X$*. Notice that the algebra of* $n$*-partitions of* $X$*, denoted by* $\wp_{\mathbf{n}}(X)$*, is isomorphic to the* $n$*BA* $\mathbf{n}^X$*.*

**Example 9.** *(*Modules over $\mathbb{Z}_m$*) Let* $m$ *be a natural number greater than* 2*, whose prime factorisation is* $m = p_1^{t_1} \dots p_k^{t_k}$*. By the Chinese Remainder Theorem (CRT, for short) the ring* $\mathbb{Z}_m$ *of integers modulo* $m$ *is isomorphic to the direct product* $\prod_{i=1}^k \mathbb{Z}_{p_i^{t_i}}$ *of the directly indecomposable rings* $\mathbb{Z}_{p_i^{t_i}}$*. By CRT each vector* $\mathbf{s} = (s_1, \dots, s_n)$ *of the* $\mathbb{Z}_m$*-module* $(\mathbb{Z}_m)^n$ *can be univocally characterised by an* $n \times k$ modular matrix $S$*, where* $S_{ij} = s_i \bmod p_j^{t_j}$*. On modular matrices, the operator* $q$*, defined in Example 3, spells out as follows:*

$$q(A, B^1, \dots, B^n)_{ij} = \sum_{l=1}^{n} A_{lj}(B^l)_{ij}, \text{ for all matrices } A, B^1, \dots, B^n.$$

*A modular matrix is* $n$*-central if and only if it is a binary matrix such that each column has at most one* 1*. It follows that the* $n$*BA of* $n$*-central modular matrices is isomorphic to the* $n$*BA* $\mathbf{n}^k$*.*

The structure theory of Boolean algebras is as good as it gets. The variety $\mathcal{BA}$ of Boolean algebras is in particular semisimple as every $\mathbf{A} \in \mathcal{BA}$ is subdirectly embeddable into a power of the 2-element Boolean algebra, which is the only subdirectly irreducible member of $\mathcal{BA}$. By and large, all these properties find some analogue in the structure theory of $n$BAs. For a start, we show that all subdirectly irreducible $n$BAs have the same finite cardinality.

**Lemma 1.** *An* $n$BA $\mathbf{A}$ *is subdirectly irreducible if and only if* $|A| = n$*.*

*Proof.* ($\Leftarrow$) We show that every $n$BA $\mathbf{A}$ with $n$ elements is simple. If $\theta$ is a congruence on $\mathbf{A}$ such that $\mathsf{e}_i \theta \mathsf{e}_j$, for some $i \neq j$, then

$$x = q(\mathsf{e}_i, \dots, x, \dots, y, \dots) \theta q(\mathsf{e}_j, \dots, x, \dots, y, \dots) = y$$

for arbitrary $x, y$. Then $\mathbf{A}$ is a simple algebra.

($\Rightarrow$) If $|A| > n$, then there exists an $a \in A$ such that $a \neq \mathsf{e}_i$ for every $i \leq n$. Since $a$ is a nontrivial $n$-central element of $\mathbf{A}$, it gives rise to a decomposition of $\mathbf{A}$ into $n \geq 2$ factors. Then $\mathbf{A}$ is directly decomposable, whence it cannot be subdirectly irreducible. □

As a direct consequence of the previous lemma, we get:

**Theorem 4.** *Any variety* $\mathcal{V}$ *of* $n$BA*s is generated by the finite set* $\{\mathbf{A} \in \mathcal{V} : |A| = n\}$*. In particular, the variety of pure* $n$BA*s is generated by the algebra* $\mathbf{n}$*.*

Notice that, if an $n$BA $\mathbf{A}$ has a minimal subalgebra $\mathbf{E}$ of cardinality $n$, then $V(\mathbf{A}) = V(\mathbf{E})$. However, we cannot assume that any two $n$-element algebras in an arbitrary variety $\mathcal{V}$ of $n$BAs are isomorphic, for such algebras may have further operations over which we do not have any control.

The next corollary shows that, for any $n \geq 2$, the $n$BA $\mathbf{n}$ plays a role analogous to the Boolean algebra $\mathbf{2}$ of truth values.

**Corollary 1.** *(i) Every $n$BA $\mathbf{A}$ is isomorphic to a subdirect product of $\mathbf{B}_1^{I_1} \times \cdots \times \mathbf{B}_k^{I_k}$ for some sets $I_1, \dots, I_k$ and some $n$BAs $\mathbf{B}_1, \dots, \mathbf{B}_k$ of cardinality $n$; (ii) Every pure $n$BA $\mathbf{A}$ is isomorphic to a subdirect power of $\mathbf{n}^I$, for some set $I$.*

*Proof.* (i) Apply Lemma 1, Theorem 4 and Birkhoff's subdirect representation theorem to the variety generated by $\mathbf{A}$. (ii) By (i) and by $\mathbf{A} \in V(\mathbf{n})$. $\square$

A subalgebra of the $n$BA $\wp_{\mathbf{n}}(X)$ of the $n$-partitions on a set $X$, defined in Example 8, is called a *field of $n$-partitions on $X$*. The Stone representation theorem for $n$BAs follows.

**Corollary 2.** *Any pure* nBA *is isomorphic to a field of $n$-partitions on a suitable set $X$.*

One of the most remarkable properties of the 2-element Boolean algebra, called *primality* in universal algebra [4, Sec. 7 in Chap. IV], is the definability of all finite Boolean functions in terms of a certain set of term operations, e.g. the connectives AND, OR, NOT. This property is inherited by $n$BAs. We prove that an algebra of cardinality $n$ is primal if and only if it is an $n$BA, so that every variety generated by an $n$-element primal algebra is a variety of $n$BAs.

**Definition 10.** *Let $\mathbf{A}$ be a nontrivial $\tau$-algebra. $\mathbf{A}$ is* primal *if it is of finite cardinality and, for every function $f : A^n \to A$ ($n \geq 0$), there is a $\tau$-term $t(x_1, \dots, x_n)$ such that for all $a_1, \dots, a_n \in A$, $f(a_1, \dots, a_n) = t^{\mathbf{A}}(a_1, \dots, a_n)$.*

A variety $\mathcal{V}$ is primal if $\mathcal{V} = V(\mathbf{A})$ for a primal algebra $\mathbf{A}$.
For all $2 \leq n < \omega$, let $\nu_n$ be the type $(q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ of pure $n$BAs.

**Definition 11.** *A $\nu_n$-term is a* head normal form *(hnf, for short) if it is defined according to the following grammar: $t, t_i ::= \mathsf{e}_i \mid x \mid q(x, t_1, \dots, t_n)$, where $x$ is an arbitrary variable. The occurrence of the variable $x$ in the hnf $t \equiv q(x, t_1, \dots, t_n)$ is called* head occurrence *of $x$ into $t$.*

**Lemma 2.** *Let $\mathbf{A}$ be a finite* $n$BA *of cardinality $n$. Then, for every function $f : A^k \to A$, there exists a canonical hnf $t$ such that $f = t^{\mathbf{A}}$.*

*Proof.* Since $A$ is an $n$BA of cardinality $n$, then $A = \{\mathsf{e}_1^{\mathbf{A}}, \dots, \mathsf{e}_n^{\mathbf{A}}\}$. We show by induction on $k$ that every function $f : A^k \to A$ is term definable in $\mathbf{A}$. If $f : A \to A$ is a unary function, then we define $f(x) = q^{\mathbf{A}}(x, f(\mathsf{e}_1^{\mathbf{A}}), f(\mathsf{e}_2^{\mathbf{A}}), \dots, f(\mathsf{e}_n^{\mathbf{A}}))$. If $f : A \times A^k \to A$ is a function of arity $k+1$, then by induction hypothesis, for each $\mathsf{e}_i^{\mathbf{A}} \in A$, there exists a term $t_i(x_1, \dots, x_k)$ such that $f(\mathsf{e}_i^{\mathbf{A}}, a_1, \dots, a_k) = t_i^{\mathbf{A}}(a_1, \dots, a_k)$ for all $a_j \in A$. Then we have: $f(a, b_1, \dots, b_k) = q^{\mathbf{A}}(a, t_1^{\mathbf{A}}(b_1, \dots, b_k), \dots, t_n^{\mathbf{A}}(b_1, \dots, b_k))$. $\square$

The following theorem is a trivial application of Lemma 2.

**Theorem 5.** *Let $\mathbf{A}$ be a finite $\tau$-algebra of cardinality $n$. Then $\mathbf{A}$ is primal if and only if it is an* $n$BA.

It follows that, if $\mathbf{A}$ is a primal algebra of cardinality $n$, then the variety generated by $\mathbf{A}$ is a variety of $n$BAs. Notice that varieties of $n$BAs generated by more than one algebra are not primal.

## 5. Applications of $n$BAs to Boolean powers

For an algebra $\mathbf{E}$ of a given type and a Boolean algebra $B$, the Boolean power of $\mathbf{E}$ by $B$ is the algebra $\mathcal{C}(B^*, \mathbf{E})$ of all continuous functions from the Stone space $B^*$ of $B$ to $E$, where $E$ is given the discrete topology and the operations of $E$ are lifted to $\mathcal{C}(B^*, \mathbf{E})$ pointwise (see, e.g., [4]). Boolean powers can be traced back to the work of Foster [7] in 1953 and turned out to be a very useful tool in universal algebra for exporting properties of Boolean algebras into other varieties.

The continuous functions from $B^*$ to $E$ determine finite partitions of $B^*$ by clopen sets. By Corollary 2 such partitions can be canonically endowed with a structure of $n$BA. Thus it is natural to algebraically rephrase the theory of Boolean powers by using central elements and, by the way, to generalise Boolean powers to arbitrary semiring powers. We define the semiring power $\mathbf{E}[R]$ of an arbitrary algebra $\mathbf{E}$ by a semiring $R$ as an algebra whose universe is the set of central elements of a certain semimodule; the operations of $\mathbf{E}$ are extended by linearity.

We obtain the following results:

(i) The central elements of an $R$-semimodule are characterised by those finite sets of elements of $R$ that are fully orthogonal, idempotent and commuting (Theorem 6).
(ii) If $R$ is a Boolean algebra, then the algebraically defined semiring power $\mathbf{E}[R]$ is isomorphic to the Boolean power $\mathcal{C}(R^*, \mathbf{E})$ (Theorem 7).
(iii) For every semiring $R$, the semiring power $\mathbf{E}[R]$ is isomorphic to the Boolean power $\mathcal{C}(C(R)^*, \mathbf{E})$ of $\mathbf{E}$ by the Boolean algebra $C(R)$ of complemented and commuting elements of $R$ (Theorem 8).

Let us now get down to the nitty-gritty.

Let $\mathbf{E}$ be an algebra of type $\tau$ and $R$ be a (non necessarily commutative) semiring. Consider the $R$-semimodule $V$ freely generated by the set $E$. A vector of $V$ is a linear combination $\mathbf{v} = \sum_{\mathsf{e}\in E} v_{\mathsf{e}}\mathsf{e}$, where $v_{\mathsf{e}} \in R$ is a scalar and $v_{\mathsf{e}} = 0$ for all but finitely many $\mathsf{e} \in E$. Every operation $g^{\mathbf{E}}$ of the algebra $\mathbf{E}$ can be linearly lifted to an operation $g^V$ on $V$.

For every finite nonempty $I = \{\mathsf{e}_1, \dots, \mathsf{e}_n\} \subseteq E$, we define:

$$q_I^V(\mathbf{v}, \mathbf{w}^1, \dots, \mathbf{w}^n) = \sum_{i=1}^n v_{\mathsf{e}_i}\mathbf{w}^i = \sum_{\mathsf{d}\in E}(\sum_{i=1}^n v_{\mathsf{e}_i} w^i_{\mathsf{d}})\mathsf{d},$$

and we call a vector $\mathbf{v} \in V$ *$I$-central* if it is an $n$-central element with respect to the operation $q_I$ in the algebra

$$V_{\mathbf{E}} = (V, g^V, q_I^V, \mathsf{e})_{g\in\tau, I\subseteq_{\mathrm{fin}} E, \mathsf{e}\in E}.$$

We now prove that a vector is $I$-central in $V_{\mathbf{E}}$ if and only if it can be written uniquely as an $R$-linear combination of orthogonal and commuting idempotents so that the sum of the idempotents is 1.

**Theorem 6.** *Let $I = \{\mathsf{e}_1, \dots, \mathsf{e}_n\} \subseteq E$. A vector $\mathbf{a} = \sum_{\mathsf{d}\in E} a_{\mathsf{d}}\mathsf{d} \in V$ is $I$-central in the algebra $V_{\mathbf{E}}$ iff the following conditions are satisfied:*

(i) $a_{\mathsf{d}} = 0$ *for all* $\mathsf{d} \in E \setminus I$*;*
(ii) $a_{\mathsf{e}_1} + \dots + a_{\mathsf{e}_n} = 1$*;*
(iii) $a_{\mathsf{e}_i}x = xa_{\mathsf{e}_i}$*, for all* $x \in R$ *and* $\mathsf{e}_i \in I$*;*
(iv) $a_{\mathsf{e}_i}a_{\mathsf{e}_j} = \begin{cases} 0 & \text{if } i \neq j \\ a_{\mathsf{e}_i} & \text{if } i = j \end{cases}$ *for all* $\mathsf{e}_i, \mathsf{e}_j \in I$*.*

*Proof.* ($\Leftarrow$) We check identities (B1)-(B3) of Theorem 3 for an element $\mathbf{a} \in V$ satisfying hypotheses (i)-(iv). Since no danger of confusion will be impending, throughout this proof we write $a_i$ for $a_{\mathsf{e}_i}$ ($\mathsf{e}_i \in I$).

B1: $q_I(\mathbf{a}, \mathsf{e}_1, \dots, \mathsf{e}_n) = a_1\mathsf{e}_1 + \cdots + a_n\mathsf{e}_n =_{(i)} \mathbf{a}$.
B2: $q_I(\mathbf{a}, \mathbf{b}, \dots, \mathbf{b}) = a_1\mathbf{b} + \cdots + a_n\mathbf{b} = (\sum_{j=1}^n a_j)\mathbf{b} =_{(ii)} 1\mathbf{b} = \mathbf{b}$.
B3:

$$\begin{array}{rcl} q_I(\mathbf{a}, \mathbf{w}^1 + \mathbf{v}^1, \dots, \mathbf{w}^n + \mathbf{v}^n) & = & \sum_{i=1}^n a_i(\mathbf{w}^i + \mathbf{v}^i) \\ & = & \sum_{i=1}^n a_i\mathbf{w}^i + \sum_{i=1}^n a_i\mathbf{v}^i \\ & = & q_I(\mathbf{a}, \mathbf{w}^1, \dots, \mathbf{w}^n) + q_I(\mathbf{a}, \mathbf{v}^1, \dots, \mathbf{v}^n) \end{array}$$

$$q_I(\mathbf{a}, r\mathbf{w}^1, \dots, r\mathbf{w}^n) = \sum_{i=1}^n a_i(r\mathbf{w}^i) = \sum_{i=1}^n (a_i r)\mathbf{w}^i =_{(iii)} \sum_{i=1}^n (ra_i)\mathbf{w}^i = rq_I(\mathbf{a}, \mathbf{w}^1, \dots, \mathbf{w}^n).$$

Without loss of generality, we assume $g \in \tau$ to be a binary operator:

$$\begin{array}{rcl} g^V(q_I(\mathbf{v}, \mathbf{t}^1, \dots, \mathbf{t}^n), q_I(\mathbf{v}, \mathbf{w}^1, \dots, \mathbf{w}^n)) & = & g^V(v_1\mathbf{t}^1 + \cdots + v_n\mathbf{t}^n, v_1\mathbf{w}^1 + \cdots + v_n\mathbf{w}^n) \\ & = & \sum_{i=1}^n \sum_{j=1}^n (v_i v_j) g^V(\mathbf{t}^i, \mathbf{w}^j) \\ & =_{(iv)} & \sum_{i=1}^n v_i g^V(\mathbf{t}^i, \mathbf{w}^i) \\ & = & q_I(\mathbf{v}, g^V(\mathbf{t}^1, \mathbf{w}^1), \dots, g^V(\mathbf{t}^n, \mathbf{w}^n)). \end{array}$$

Let $\mathbf{c}^1, \dots, \mathbf{c}^n \in V$, $J = \{\mathsf{d}_1, \dots, \mathsf{d}_k\} \subseteq E$, and $Y$ be a $k \times n$ matrix of vectors of $V$. We denote by $Y^j$ the $j$-th column of $Y$ and by $Y_j$ its $j$-th row. If $1 \le s \le n$ then we have: $q_J(\mathbf{c}^s, Y^s) = q_J(\mathbf{c}^s, Y_1^s, \dots, Y_k^s) = \sum_{l=1}^k c_{\mathsf{d}_l}^s Y_l^s$ and

$$q_I(\mathbf{a}, q_J(\mathbf{c}^1, Y^1), \dots, q_J(\mathbf{c}^n, Y^n)) = \sum_{s=1}^n a_s q_J(\mathbf{c}^s, Y^s) = \sum_{l=1}^k \sum_{s=1}^n (a_s c_{\mathsf{d}_l}^s) Y_l^s.$$

As $q_I(\mathbf{a}, \mathbf{c}^1, \dots, \mathbf{c}^n) = \sum_{j=1}^n a_j \mathbf{c}^j = \sum_{j=1}^n a_j (\sum_{\mathsf{h} \in E} c_{\mathsf{h}}^j \mathsf{h}) = \sum_{\mathsf{h} \in E} (\sum_{j=1}^n a_j c_{\mathsf{h}}^j)\mathsf{h}$ and $J = \{\mathsf{d}_1, \dots, \mathsf{d}_k\}$, we get the conclusion by applying (iii)-(iv):

$$\begin{array}{rcl} q_J(q_I(\mathbf{a}, \mathbf{c}^1, \dots, \mathbf{c}^n), q_I(\mathbf{a}, Y_1), \dots, q_I(\mathbf{a}, Y_k)) & = & \sum_{l=1}^k q_I(\mathbf{a}, \mathbf{c}^1, \dots, \mathbf{c}^n)_{\mathsf{d}_l} q_I(\mathbf{a}, Y_l) \\ & = & \sum_{l=1}^k ([\sum_{j=1}^n a_j c_{\mathsf{d}_l}^j][\sum_{s=1}^n a_s Y_l^s]) \\ & = & \sum_{l=1}^k (\sum_{s=1}^n \sum_{j=1}^n a_j c_{\mathsf{d}_l}^j a_s) Y_l^s \\ & = & \sum_{l=1}^k \sum_{s=1}^n (a_s c_{\mathsf{d}_l}^s) Y_l^s \\ & = & q_I(\mathbf{a}, q_J(\mathbf{c}^1, Y^1), \dots, q_J(\mathbf{c}^n, Y^n)). \end{array}$$

($\Rightarrow$) Since $\mathbf{a}$ is $I$-central, then by (B1) we have that $\mathbf{a} = q(\mathbf{a}, \mathsf{e}_1, \dots, \mathsf{e}_n) = \sum_{i=1}^n a_i \mathsf{e}_i$. This implies (i). Item (ii) follows by (B2) and by (i), because $\mathsf{e}_1 = q_I(\mathbf{a}, \mathsf{e}_1, \dots, \mathsf{e}_1) = (\sum_{i=1}^n a_i)\mathsf{e}_1$.

To show (iii) we consider the following chain of equalities: $(ra_1)\mathsf{e}_1 = r(a_1\mathsf{e}_1) = rq_I(\mathbf{a}, \mathsf{e}_1, \mathbf{0}, \dots, \mathbf{0}) =_{(B3)} q_I(\mathbf{a}, r\mathsf{e}_1, r\mathbf{0} \dots, r\mathbf{0}) = a_1(r\mathsf{e}_1) = (a_1 r)\mathsf{e}_1$. It follows that $ra_1 = a_1 r$ for every $r \in R$. Similarly for the other coordinates of $\mathbf{a}$.

From (B3)

$$q_I(\mathbf{a}, q_I(\mathbf{c}^1, Y^1), \dots, q_I(\mathbf{c}^n, Y^n)) = q_I(q_I(\mathbf{a}, \mathbf{c}^1, \dots, \mathbf{c}^n), q_I(\mathbf{a}, Y_1), \dots, q_I(\mathbf{a}, Y_n))$$

it follows that

$$\sum_{s=1}^n a_s \left( \sum_{l=1}^n c_l^s Y_l^s \right) = \sum_{l=1}^n \sum_{s=1}^n (a_s c_l^s) Y_l^s =_{\mathrm{B3}} \sum_{l=1}^n \sum_{s=1}^n (\sum_{j=1}^n a_j c_l^j a_s) Y_l^s.$$

Fix $\mathsf{e} \in I$, a row $l$ and column $s$. Let $Y_l^s = \mathbf{e}$ and $Y_r^i = \mathbf{0}$ for all $(i, r) \neq (s, l)$. Then

$$a_s c_l^s = \sum_{j=1}^n a_j c_l^j a_s.$$

We get $a_s = a_s a_s$ of item (iv) by putting $c_l^s = 1$ and $c_l^j = 0$ for all $j \neq s$. The last condition $a_j a_s = 0$ $(j \neq s)$ is obtained by defining $c_l^s = 0$, $c_l^j = 1$ and $c_l^i = 0$ for all $i \neq j$. □

**Definition 12.** *A vector* $\mathbf{v} \in V$ *is called* finitely central *if* $\mathbf{v}$ *is* $I$*-central for some nonempty finite subset* $I$ *of* $E$*. We denote by* $E[R]$ *the set of all finitely central elements of* $V_{\mathbf{E}}$*.*

**Lemma 3.** (i) *The set* $E[R]$ *is a subuniverse of the algebra* $(V, g^V)_{g\in\tau}$*.*

(ii) *If* $E$ *has finite cardinality* $n$*, then* $E[R]$ *is closed under the operation* $q_E^V$ *and the algebra* $(E[R], g^V, q_E^V, \mathsf{e})_{\mathsf{e}\in E, g\in\tau}$ *is an* $n$BA*.*

*Proof.* (i) Without loss of generality, assume $g \in \tau$ to be a binary operator. Let $\mathbf{v}$ be $I$-central and $\mathbf{t}$ be $J$-central (for $I, J$ finite subsets of $E$). Then we set:

$$\mathbf{w} = g^V(\mathbf{v}, \mathbf{t}) = \sum_{\mathsf{d},\mathsf{e}\in E} (v_{\mathsf{d}} t_{\mathsf{e}}) g^{\mathbf{E}}(\mathsf{d}, \mathsf{e}). \tag{5.1}$$

We show that $\mathbf{w}$ is $H$-central, where $H = \{g^{\mathbf{E}}(\mathsf{d}, \mathsf{e}) : \mathsf{d} \in I, \mathsf{e} \in J\}$. Let $H_{\mathsf{c}} = \{(\mathsf{d}, \mathsf{e}) \in I \times J : g^{\mathbf{E}}(\mathsf{d}, \mathsf{e}) = \mathsf{c}\}$. Then by (5.1) $w_{\mathsf{c}} = \sum_{(\mathsf{d},\mathsf{e})\in H_{\mathsf{c}}} v_{\mathsf{d}} t_{\mathsf{e}}$ is the $\mathsf{c}$-coordinate of $\mathbf{w}$. The conclusion follows from Theorem 6 by verifying that $w_{\mathsf{d}} = 0$ for $\mathsf{d} \notin H$, $\sum_{\mathsf{e}\in H} w_{\mathsf{e}} = 1$, $w_{\mathsf{d}} w_{\mathsf{d}} = w_{\mathsf{d}}$, $w_{\mathsf{d}} w_{\mathsf{e}} = 0$ for $\mathsf{d} \neq \mathsf{e}$, and $\forall x \in R. w_{\mathsf{e}} x = x w_{\mathsf{e}}$. In the proof we use the corresponding properties of the vectors $\mathbf{v}$ and $\mathbf{t}$.

(ii) If an element $\mathbf{v}$ is $I$-central, then by Theorem 6 $\mathbf{v}$ is also $J$-central for every finite $J$ such that $I \subseteq J \subseteq E$. Then the conclusion follows because $E$ is finite and all elements of $E[R]$ are $E$-central, so that $E[R]$ is the set of all $E$-central elements of the algebra $(V, g^V, q_E, \mathsf{e})_{g\in\tau,\mathsf{e}\in E}$. □

**Definition 13.** *The algebra* $\mathbf{E}[R] = (E[R], g^{\mathbf{E}[R]})_{g\in\tau}$*, called the* semiring power *of* $\mathbf{E}$ *by* $R$*, is the algebra of finitely central elements of the free* $R$*-semimodule generated by* $\mathbf{E}$*.*

If $R$ is a Boolean algebra and $\mathbf{E}$ an arbitrary algebra, let $\mathcal{C}(R^*, \mathbf{E})$ be the set of continuous functions from the Stone space $R^*$ to $E$, giving $E$ the discrete topology. $\mathcal{C}(R^*, \mathbf{E})$ is a subuniverse of the algebra $\mathbf{E}^{R^*}$, called the *Boolean power* of $\mathbf{E}$ by $R$ [4].

**Theorem 7.** *If* $R$ *is a Boolean algebra, then* $\mathbf{E}[R]$ *is isomorphic to the Boolean power* $\mathcal{C}(R^*, \mathbf{E})$*.*

*Proof.* Let $\mathbf{v} = \sum_{\mathsf{e}_i\in I} v_{\mathsf{e}_i} \mathsf{e}_i \in E[R]$ for some $I \subseteq_{\text{fin}} E$. Given an ultrafilter $F \in R^*$, by Theorem 6 there exists exactly one $\mathsf{e}_i \in I$ such that $v_{\mathsf{e}_i} \in F$. Then we define $f_{\mathbf{v}} : R^* \to E$ as follows, for every $F \in R^*$:

$$f_{\mathbf{v}}(F) = \text{ the unique } \mathsf{e}_i \in I \text{ such that } v_{\mathsf{e}_i} \in F.$$

The map $\mathbf{v} \mapsto f_{\mathbf{v}}(F)$ is an isomorphism from $\mathbf{E}[R]$ to $\mathcal{C}(R^*, \mathbf{E})$. □

In the remaining part of this section we prove that every semiring power is isomorphic to a Boolean power.

Let $R$ be a semiring. An element $r \in R$ is (i) *complemented* if $(\exists s)\ r + s = 1$ and $rs = 0$; (ii) *commuting* if $(\forall t)\ rt = tr$. The complement of an element $r$ is unique, and it is denoted by $r'$. Indeed, if $s$ is another complement of $r$ then we have: $r' = r'(r+s) = r's$ and $s = (r+r')s = r's$.

We denote by $C(R)$ the set of complemented and commuting elements of $R$. Every element of C$(R)$ is idempotent, because $r = r(r + r') = r^2 + rr' = r^2$.

In the following lemma we show that $C(R)$ is a Boolean algebra with respect to the operations $r \vee s = r + r's$, $r \wedge s = rs$ and the above defined complementation.

**Lemma 4.** $C(R)$ *is a Boolean algebra.*

*Proof.*

- $C(R)$ is closed under complementation. If $r \in$ C$(R)$, then we prove that $r'$ is commuting: $r't = r't(r+r') = r'tr+r'tr' = r'rt+r'tr' = 0+r'tr' = r'tr'$. By symmetry we also have $tr' = r'tr'$.

- $C(R)$ is closed under $\vee$. If $r, s \in \mathrm{C}(R)$, then we prove that $r's'$ is the complement of $r \vee s$: $(r \vee s) + r's' = r + r's + r's' = r + r'(s + s') = r + r' = 1$ and $(r \vee s)r's' = (r + r's)r's' = rr's' + r'sr's' = 0 + 0 = 0$, by commutativity. Moreover, it is not difficult to check that $r \vee s$ is commuting.
- Idempotence: $r \vee r = r + r'r = r + 0 = r$;
- De Morgan Laws: $(r \vee s)' = r's'$ and $(rs)' = r' + rs'$, because $(r + r's)r's' = rr's' + r'sr's' = 0 + r'^2ss' = 0$ and $r + r's + r's' = r + r'(s + s') = r + r' = 1$;
- Commutativity: $r \vee s = r + r's = (s + s')(r + r's) = sr + s'r + sr's + s'r's = sr + s'r + sr's = s(r + r') + s'r = s + s'r = s \vee r$.
- Associativity: $(r \vee s) \vee t = r + r's + (r \vee s)'t = r + r's + r's't = r + r'(s + s't) = r \vee (s \vee t)$.

We leave to the reader the verification of the other laws. $\square$

**Lemma 5.** *Let $a_1, \dots, a_n \in C(R)$.*

(a) $(a_1 + \cdots + a_n)(a_1 \vee \cdots \vee a_n) = a_1 + \cdots + a_n$.
(b) *If $a_i a_j = 0$ $(i \neq j)$, then $a_1 + \cdots + a_n = a_1 \vee \cdots \vee a_n$.*

*Proof.* (a) $(\sum_{i=1}^n a_i)(\bigvee_{i=1}^n a_i) = \sum_{i=1}^n (a_i(\bigvee_{i=1}^n a_i)) = \sum_{i=1}^n (a_i \wedge (\bigvee_{i=1}^n a_i)) = \sum_{i=1}^n a_i$.

(b) The proof is by induction.

$(n = 2)$: $a_1 \vee a_2 = (a_1 \vee a_2) + a_1 a_2 = a_1 + a_1' a_2 + a_1 a_2 = a_1 + (a_1 + a_1')a_2 = a_1 + a_2$.

$(n > 2)$: $a_1 \vee a_2 \vee \cdots \vee a_n = a_1 \vee (\bigvee_{j=2}^n a_j) = a_1 \vee (\sum_{j=2}^n a_j) = a_1 + (\sum_{j=2}^n a_j) = a_1 + a_2 + \cdots + a_n$ because $a_1(\sum_{j=2}^n a_j) = \sum_{j=2}^n a_1 a_j = 0$. $\square$

**Theorem 8.** *Let $R$ be a semiring and $\mathbf{E}$ be an algebra. The semiring power $\mathbf{E}[R]$ is isomorphic to the Boolean power $\mathcal{C}(C(R)^*, \mathbf{E})$.*

*Proof.* By Theorem 7 it is sufficient to prove that $\mathbf{E}[R]$ coincides with $\mathbf{E}[C(R)]$.

Let $\mathbf{v} = \sum_{i=1}^n v_i \mathsf{e}_i \in E[R]$. By Theorem 6 the coordinates $v_1, \dots, v_n$ are idempotent, commuting and orthogonal, i.e., $\sum_{i=1}^n v_i = 1$ and $v_i v_j = 0$ $(i \neq j)$. Since $v_i + \sum_{j \neq i} v_j = 1$ and $v_i(\sum_{j \neq i} v_j) = 0$, then we get $v_i \in C(R)$, for every $i = 1, \dots, n$. By $v_1 + \cdots + v_n = 1$ and by Lemma 5(a) we derive $v_1 \vee \cdots \vee v_n = 1$. Thus $\mathbf{v} \in E[C(R)]$.

For the converse, let $\mathbf{v} = \sum_{i=1}^n v_i \mathsf{e}_i \in E[C(R)]$. Then the coordinates $v_1, \dots, v_n$ are idempotent, commuting and orthogonal in the Boolean algebra $C(R)$, i.e., $\bigvee_{i=1}^n v_i = 1$ and $v_i \wedge v_j = v_i v_j = 0$ $(i \neq j)$. By Lemma 5(b) we derive $v_1 + \cdots + v_n = 1$. Thus, $\mathbf{v} \in E[R]$.

The operations on $\mathbf{E}[R]$ and those on $\mathbf{E}[C(R)]$ coincide. $\square$

## 6. Representation Theorems

In this section we present two representation theorems:

(i) Any pure $n$BA is isomorphic to the algebra of $n$-central elements of a certain Boolean vector space, namely, the free $n$-generated Boolean vector space over the Boolean algebra of 2-central elements of $\mathbf{A}$ (Theorem 9).
(ii) Any member of a variety $V(\mathbf{A})$ generated by an $n$BA $\mathbf{A}$ of cardinality $n$ is isomorphic to a Boolean power of the generator $\mathbf{A}$ (Theorem 10). A notable consequence of this result and of Theorem 5 is Foster's Theorem for primal varieties (cfr. [4, Thm. 7.4]).

The main technical tool is the definition of a Boolean algebra $B_{\mathbf{A}}$ living inside every $n$BA $\mathbf{A}$. We use $B_{\mathbf{A}}$ to define the coordinates $a_1, \dots, a_n \in B_{\mathbf{A}}$ of every element $a \in A$. The map $a \mapsto (a_1, \dots, a_n)$ provides the embedding of $\mathbf{A}$ into the $n$BA of $n$-central elements of the Boolean vector space $B_{\mathbf{A}}^n$.

6.1. **The inner Boolean algebra of an $n$BA.** Any $n$DA $\mathbf{A}$ accommodates within itself algebras of dimension $m$, for any $m < n$. Indeed, if $\mathbf{A}$ is an $n$DA, and $m < n$, set

$$p(a, a_1, \dots, a_m) = q(a, a_1, \dots, a_m, \mathsf{e}_{m+1}, \dots, \mathsf{e}_n). \tag{6.1}$$

It is straightforward to verify that $\mathbf{A}$ is an $m$DA w.r.t. the defined $p$ and $\mathsf{e}_1, \dots, \mathsf{e}_m$. A tedious but easy computation shows the following lemma.

**Lemma 6.** *Let $x$ be an $n$-central element of $\mathbf{A}$. Then $x$ is $m$-central iff $p(x, y, y, \dots, y) = y$, for all $y \in A$.*

Let $\mathbf{A}$ be an $n$BA. The set $B_{\mathbf{A}} = \{x \in A \ : \ p(x, y, y) = y\}$ of the 2-central elements of $\mathbf{A}$ with respect to the ternary term operation $p$ and constants $\mathsf{e}_1, \mathsf{e}_2$ is a Boolean algebra (see [20]).

**Definition 14.** *Let $\mathbf{A}$ be an $n$BA. The Boolean algebra $B_{\mathbf{A}}$, whose operations are defined as follows:*

$$x \wedge y = p(x, \mathsf{e}_1, y); \quad x \vee y = p(x, y, \mathsf{e}_2); \quad \neg x = p(x, \mathsf{e}_2, \mathsf{e}_1); \quad 0 = \mathsf{e}_1; \quad 1 = \mathsf{e}_2,$$

*is called the* Boolean algebra of the coordinates of $\mathbf{A}$.

As a matter of notation, we write $q(x, y, z, \bar{u})$ for $q(x, y, z, u, \dots, u)$. The next lemma gathers some useful properties of the algebra $B_{\mathbf{A}}$.

**Lemma 7.** *Let $\mathbf{A}$ be an $n$BA, $y, x_1, \dots, x_n \in B_{\mathbf{A}}$ and $a \in A$. Then we have:*

(i) $q(y, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) = y$.
(ii) $q(a, x_1, \dots, x_n) \in B_{\mathbf{A}}$.
(iii) $B_{\mathbf{A}} = \{q(a, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) \ : \ a \in A\}$.

*Proof.* (i)

$$\begin{array}{rcll} q(y, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) & = & q(y, \mathsf{e}_1, \mathsf{e}_2, p(y, \mathsf{e}_1, \mathsf{e}_1), \dots, p(y, \mathsf{e}_1, \mathsf{e}_1)) & \text{by } y \in B_{\mathbf{A}} \\ & = & q(y, \mathsf{e}_1, \mathsf{e}_2, q(y, \mathsf{e}_1, \mathsf{e}_1, \mathsf{e}_3, \dots, \mathsf{e}_n), \dots, q(y, \mathsf{e}_1, \mathsf{e}_1, \mathsf{e}_3, \dots, \mathsf{e}_n)) & \text{by Def. of } p \\ & = & q(y, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \dots, \mathsf{e}_n) & \text{by (B4)} \\ & = & y. & \end{array}$$

(ii) Let $a \in A$ and $\overline{x} = x_1, \dots, x_n \in B_{\mathbf{A}}$. As $a$ is $n$-central, then by Lemma 6 $q(a, \overline{x}) \in B_{\mathbf{A}}$ iff $p(q(a, \overline{x}), z, z) = z$, for any $z \in A$.

$$\begin{array}{rcll} p(q(a, \overline{x}), z, z) & = & q(q(a, \overline{x}), z, z, \mathsf{e}_3, \dots, \mathsf{e}_n) & \\ & = & q(a, \dots, q(x_i, z, z, \mathsf{e}_3, \dots, \mathsf{e}_n), \dots) & \text{by (B3)} \\ & = & q(a, \dots, p(x_i, z, z), \dots) & \text{by Def. of } p \\ & = & q(a, z, \dots, z, \dots, z) & \text{by (B2)} \\ & = & z & \text{by (B2)} \end{array}$$

(iii) By (i)-(ii) applied to $q(a, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1)$. $\square$

6.2. **The coordinates of an element.** Let $\mathbf{A}$ be an $n$BA, and $\sigma$ be a permutation of $1, \dots, n$. For any $a \in A$, we write $a^\sigma$ for $q(a, \mathsf{e}_{\sigma 1}, \dots, \mathsf{e}_{\sigma n})$. In particular, if $(2i)$ is the transposition defined by $(2i)(2) = i$, $(2i)(i) = 2$ and $(2i)(k) = k$ for $k \neq 2, i$, then we have

$$a^{(2i)} = q(a, \mathsf{e}_{(2i)1}, \dots, \mathsf{e}_{(2i)n}) = q(a, \mathsf{e}_1, \mathsf{e}_i, \mathsf{e}_3, \dots, \mathsf{e}_{i-1}, \mathsf{e}_2, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n).$$

We write $q(a, \mathsf{e}_i/2; \mathsf{e}_2/i)$ for $a^{(2i)}$.

**Definition 15.** *Let $(B_{\mathbf{A}})^n$ be the Boolean vector space of dimension $n$ over $B_{\mathbf{A}}$. The* vector of the coordinates *of an element $a \in A$ is a tuple $(a_1, \dots, a_n) \in (B_{\mathbf{A}})^n$, where*

$$a_i = q(a^{(2i)}, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1).$$

Observe that by Lemma 7(ii) $a_i \in B_{\mathbf{A}}$ for every $i$. The next lemma shows that the coordinate $a_i$ admits a simpler description.

**Lemma 8.** *If* $\mathbf{A}$ *is an* $n$BA *and* $a \in A$, *then* $a_i = q(a, \mathsf{e}_1, \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1)$, *where* $\mathsf{e}_2$ *is at position* $i$.

*Proof.*

$$\begin{array}{rcll} a_i & = & q(a^{(2i)}, \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) & \text{by Def.} \\ & = & q(q(a, \mathsf{e}_i/2; \mathsf{e}_2/i), \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) & \\ & = & q(q(a, \mathsf{e}_1, \mathsf{e}_i, \mathsf{e}_3, \dots, \mathsf{e}_{i-1}, \mathsf{e}_2, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n), \mathsf{e}_1, \mathsf{e}_2, \bar{\mathsf{e}}_1) & \\ & = & q(a, \mathsf{e}_1, \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1) & \mathsf{e}_2 \text{ at position } i \\ & = & q(a, \mathsf{e}_2/i; \mathsf{e}_1/\bar{\imath}). & \end{array}$$

□

**Example 10.** *By Theorem 6 an* $n$*-subset of* $X$ *(see Example 8) is* $n$*-central iff it is an* $n$*-partition of* $X$. *If* $P = (P_1, \dots, P_n)$ *is an* $n$*-partition of* $X$, *then the* $i$*-th coordinate of* $P$ *is* $(X \setminus P_i, P_i, \emptyset, \dots, \emptyset)$.

The following lemma follows directly from the definition.

**Lemma 9.** *If* $a \in B_{\mathbf{A}}$, *then* $a_1 = \neg a$; $\quad a_2 = a$; $\quad a_k = 0$ $(3 \leq k \leq n)$.

The coordinates of the result of an application of $q$ to elements of $A$ can be expressed as the result of an application of the Boolean operations of $B_{\mathbf{A}}$ to the coordinates of the arguments.

**Lemma 10.** *Let* $\mathbf{A}$ *be an* $n$BA. *For every* $a, b^1, \dots, b^n \in A$ *we have:*

(i) *The coordinates of* $a$ *are fully orthogonal, i.e.,* $\bigvee_{i=1}^n a_i = 1$ *and* $a_i \wedge a_k = 0$, *for every* $i \neq k$.
(ii) $q(a, b^1, \dots, b^n)_i = q(a, (b^1)_i, \dots, (b^n)_i) = \bigvee_{j=1}^n a_j \wedge (b^j)_i$, *where* $a_j$ *is the* $j$*-th coordinate of* $a$ *and* $(b^j)_i$ *is the* $i$*-th coordinate of* $b^j$.

*The join* $\bigvee$ *and the meet* $\wedge$ *are taken in the Boolean algebra* $B_{\mathbf{A}}$.

*Proof.* It suffices to check the previous identities in any $n$-element generator of the variety generated by $\mathbf{A}$. □

**Lemma 11.** *If* $a, b \in A$ *have the same coordinates, i.e.* $a_i = b_i$ *for all* $i$, *then* $a = b$.

*Proof.* Since in the definition of vector of coordinates we only use the $q$ operator and the constants, we can safely restrict ourselves to the pure reduct of $\mathbf{A}$. By Corollary 1 the pure reduct of $\mathbf{A}$ is a subalgebra of $\mathbf{n}^I$, for an appropriate $I$. It is routine to verify in $\mathbf{n}^I$ that if two elements have the same coordinates then they coincide, whence the same is true for $\mathbf{A}$. □

6.3. **The main theorems.** We recall that an algebra $\mathbf{A}$ is a retract of an algebra $\mathbf{B}$, and we write $\mathbf{A} \lhd \mathbf{B}$, if there exist two homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{B} \to \mathbf{A}$ such that $g \circ f = \mathrm{Id}_A$.

**Theorem 9.** *Let* $\mathbf{A}$ *be a pure* $n$BA *and* $\mathrm{Ce}(B^n_{\mathbf{A}})$ *be the* $n$BA *of* $n$*-central elements of the Boolean vector space* $B^n_{\mathbf{A}}$. *Then we have:*

$$\mathbf{A} \cong \mathrm{Ce}(B^n_{\mathbf{A}}) \cong \mathbf{n}[B_{\mathbf{A}}] \lhd \mathbf{A}[B_{\mathbf{A}}].$$

*Proof.* We prove that $\mathbf{A} \cong \mathbf{n}[B_{\mathbf{A}}]$. Define the map $f : \mathbf{A} \to \mathbf{n}[B_{\mathbf{A}}]$ as follows, for any $a \in A$:

$$f(a) = \sum_{i=1}^n a_i \mathsf{e}_i,$$

where $a_1,\dots,a_n$ are the coordinates of $a$. The map $f$ is injective by Lemma 11 and it preserves the operation $q$:

$$\begin{array}{rcll}
q^{\mathbf{n}[B_{\mathbf{A}}]}(f(a),f(b^1),\dots,f(b^n)) &=& q^{\mathbf{n}[B_{\mathbf{A}}]}(\sum_{i=1}^n a_i\mathsf{e}_i,\sum_{i=1}^n b_i^1\mathsf{e}_i,\dots,\sum_{i=1}^n b_i^n\mathsf{e}_i) & \\
&=& a_1(\sum_{i=1}^n b_i^1\mathsf{e}_i)+\dots+a_n(\sum_{i=1}^n b_i^n\mathsf{e}_i) & \\
&=& \sum_{i=1}^n(a_1\wedge b_i^1)\mathsf{e}_i+\dots+\sum_{i=1}^n(a_n\wedge b_i^n)\mathsf{e}_i & \\
&=& \sum_{i=1}^n\left((a_1\wedge b_i^1)\vee\dots\vee(a_n\wedge b_i^n)\right)\mathsf{e}_i & \\
&=& \sum_{i=1}^n q^{\mathbf{A}}(a,b^1,\dots,b^n)_i\mathsf{e}_i & \text{by Lem. 10(ii)} \\
&=& f(q^{\mathbf{A}}(a,b^1,\dots,b^n)). &
\end{array}$$

Next we prove that $f$ is surjective. Let $\mathbf{a}=\sum_{i=1}^n a_i\mathsf{e}_i\in\mathbf{n}[B_{\mathbf{A}}]$. Recall from Section 6.1 the definition of the ternary operator $p$. We will show that $f(b)=\mathbf{a}$, where

$$b=p(a_1,p(a_2,p(a_3,p(\dots p(a_{n-1},\mathsf{e}_n,\mathsf{e}_{n-1}))\dots),\mathsf{e}_3),\mathsf{e}_2),\mathsf{e}_1).$$

We write $b^i=p(a_i,p(a_{i+1},p(a_{i+2},p(\dots p(a_{n-1},\mathsf{e}_n,\mathsf{e}_{n-1}))\dots),\mathsf{e}_{i+2}),\mathsf{e}_{i+1}),\mathsf{e}_i)$ in such a way that $b=b^1$.

As $\bigvee_{i=1}^n a_i=1$ and $a_ia_j=0$ $(i\neq j)$, we have that the complement $\neg a_i$ of $a_i$ is equal to $\bigvee_{j\neq i}a_j$. For an arbitrary $x\in A$, we have:

$$\begin{array}{rcll}
p(a_i,x,\mathsf{e}_i)_k &=& q(a_i,x,\mathsf{e}_i,\mathsf{e}_3,\dots,\mathsf{e}_n)_k & \\
&=& ((a_i)_1\wedge x_k)\vee((a_i)_2\wedge(\mathsf{e}_i)_k) & \text{by Lemma 9 and Lemma 10(ii)} \\
&=& (\neg a_i\wedge x_k)\vee(a_i\wedge(\mathsf{e}_i)_k) &
\end{array}$$

Since $\mathsf{e}_1=0$ is the bottom and $\mathsf{e}_2=1$ is the top of $B_{\mathbf{A}}$, then we obtain:

$$p(a_i,x,\mathsf{e}_i)_k=\begin{cases}(\neg a_i\wedge x_i)\vee(a_i\wedge\mathsf{e}_2)=(\neg a_i\wedge x_i)\vee a_i & \text{if } k=i\\ (\neg a_i\wedge x_k)\vee(a_i\wedge\mathsf{e}_1)=\neg a_i\wedge x_k & \text{if } k\neq i\end{cases}$$

If the coordinate $x_j$ is equal to 0 for every $j\leq i$, then we derive:

$$(\forall j\leq i.\ x_j=0)\ \Rightarrow\ p(a_i,x,\mathsf{e}_i)_k=\begin{cases}a_i & \text{if } k=i\\ 0 & \text{if } k<i\\ \neg a_i\wedge x_k & \text{if } k>i\end{cases}$$

It follows that

$$b^{n-1}=p(a_{n-1},\mathsf{e}_n,\mathsf{e}_{n-1})_k=\begin{cases}a_{n-1} & \text{if } k=n-1\\ 0 & \text{if } k<n-1\\ \neg a_{n-1}\wedge(\mathsf{e}_n)_n=\neg a_{n-1}\wedge\mathsf{e}_2=\neg a_{n-1} & \text{if } k=n\end{cases}$$

By iterating we get

$$b^i=\begin{cases}a_j & \text{if } j=i,\dots,n-1\\ 0 & \text{if } k<i\\ \neg\bigvee_{j=i}^{n-1}a_j & \text{if } k=n\end{cases}$$

Hence, recalling that $b=b^1$, we have the conclusion $f(b)=\mathbf{a}$.

The definitions of the isomorphism $\mathrm{Ce}(B_{\mathbf{A}}^n)\cong\mathbf{n}[B_{\mathbf{A}}]$ and of the retraction $\mathbf{n}[B_{\mathbf{A}}]\lhd\mathbf{A}[B_{\mathbf{A}}]$ are straightforward. □

We are now ready to extend the above result to any similarity type.

**Theorem 10.** *Let* $\mathbf{A}$ *be an* $n$BA *of type* $\tau$*, whose minimal subalgebra* $\mathbf{E}$ *has finite cardinality* $n$*. Then we have:*

$$\mathbf{A}\cong\mathbf{E}[B_{\mathbf{A}}]\cong\mathcal{C}(B_{\mathbf{A}}^*,\mathbf{E}).$$

*Proof.* As the algebra $\mathbf{E}$ is an expansion of the $n$BA $\mathbf{n}$ by operations of the type $\tau$, it is sufficient to prove that the map $f$, defined in the proof of Theorem 9, is a homomorphism for the type $\tau$. The other isomorphism follows from Theorem 7.

Let now $g \in \tau$ be an operator that is supposed to be binary to avoid unnecessary notational issues. We observe that, for $a, b \in A$,

$$g^{\mathbf{E}[B_{\mathbf{A}}]}(f(a), f(b)) = g^{\mathbf{E}[B_{\mathbf{A}}]}(\sum_{i=1}^{n} a_i \mathsf{e}_i, \sum_{j=1}^{n} b_j \mathsf{e}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i \wedge b_j) g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j) = \sum_{k=1}^{n} w_k \mathsf{e}_k,$$

where

$$w_k = \bigvee_{\{(i,j): g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j) = \mathsf{e}_k\}} a_i \wedge b_j.$$

We have to show that $f(g^{\mathbf{A}}(a,b))$ is equal to $\sum_{k=1}^{n} w_k \mathsf{e}_k$, that is, $g^{\mathbf{A}}(a,b)_k = w_k$ for every $1 \le k \le n$. Note that by hypothesis the variety $V(\mathbf{A})$ generated by $\mathbf{A}$ coincides with the variety $V(\mathbf{E})$ generated by $\mathbf{E}$, and by Birkhoff's Theorem there exists a set $I$ such that $\mathbf{A}$ is isomorphic to a subdirect product of $\mathbf{E}^I$. Then it is sufficient to prove that, for every $h \in I$, $w_k(h) = g^{\mathbf{A}}(a,b)_k(h)$. Since both $w_k$ and $g^{\mathbf{A}}(a,b)_k$ belong to $B_{\mathbf{A}}$, then both $w_k(h)$ and $g^{\mathbf{A}}(a,b)_k(h)$ are elements of $B_{\mathbf{E}} = \{\mathsf{e}_1, \mathsf{e}_2\}$, where $0 = \mathsf{e}_1$ and $1 = \mathsf{e}_2$.

By definition of $w_k = \bigvee_{\{(i,j): g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j) = \mathsf{e}_k\}} (a_i \wedge b_j)$ the $h$-th component $w_k(h)$ of $w_k$ is defined as follows:

$$w_k(h) = \begin{cases} \mathsf{e}_2 & \text{if } \exists ij \ g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j) = \mathsf{e}_k \text{ and } a_i(h) = b_j(h) = \mathsf{e}_2 \\ \mathsf{e}_1 & \text{otherwise.} \end{cases}$$

On the other hand, by Lemma 8 $g^{\mathbf{A}}(a,b)_k = q(g^{\mathbf{A}}(a,b), \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1)$, where $\mathsf{e}_2$ is at $k$-position, so that

$$g^{\mathbf{A}}(a,b)_k(h) = \begin{cases} \mathsf{e}_2 & \text{if } g^{\mathbf{A}}(a,b)(h) = \mathsf{e}_k \\ \mathsf{e}_1 & \text{otherwise.} \end{cases}$$

Note that $g^{\mathbf{A}}(a,b)(h) = g^{\mathbf{E}}(a(h), b(h)) = g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j)$, for some $i, j$. Then $g^{\mathbf{A}}(a,b)(h) = \mathsf{e}_k$ if and only if, for some $i$ and $j$, $a(h) = \mathsf{e}_i$, $b(h) = \mathsf{e}_j$ and $g^{\mathbf{E}}(\mathsf{e}_i, \mathsf{e}_j) = \mathsf{e}_k$. Then it suffices to prove that $a(h) = \mathsf{e}_i$ and $b(h) = \mathsf{e}_j$ if and only if $a_i(h) = \mathsf{e}_2$ and $b_j(h) = \mathsf{e}_2$. This follows from

$$\begin{array}{rcll} a_i(h) & = & q(a, \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1)(h) & \mathsf{e}_2 \text{ at } i\text{-position} \\ & = & q(a(h), \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1) & \\ & = & q(\mathsf{e}_i, \mathsf{e}_1, \dots, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1, \dots, \mathsf{e}_1) & \\ & = & \mathsf{e}_2 & \end{array}$$

and, similarly, $b_j(h) = \mathsf{e}_2$. □

As a corollary of the previous results, we obtain Foster's Theorem for primal algebras:

**Corollary 3.** *If $\mathbf{P}$ is a primal algebra of cardinality $n$, then any $\mathbf{A} \in V(\mathbf{P})$ is isomorphic to the Boolean power $\mathcal{C}(B^*_{\mathbf{A}}, \mathbf{P})$, for the Boolean algebra $B_{\mathbf{A}}$ defined in Section 6.1.*

*Proof.* By Theorem 5 $\mathbf{P}$ is an $n$BA. If $\mathbf{A} \in V(\mathbf{P})$, then the minimal subalgebra of $\mathbf{A}$ coincides with $\mathbf{P}$ itself, because the constants $\mathsf{e}_1, \dots, \mathsf{e}_n$ belongs to $A$ and are closed under the operations of the algebra. By Theorem 10 $\mathbf{A}$ is isomorphic to $\mathcal{C}(B^*_{\mathbf{A}}, \mathbf{P})$. □

## 7. Applications to logic

For all $2 \leq n < \omega$, let $\nu_n$ be the type $(q, \mathsf{e}_1, \ldots, \mathsf{e}_n)$, where $q$ is $n+1$-ary and, for every $i \leq n$, $\mathsf{e}_i$ is nullary. For every natural number $n \geq 2$, the algebra $\mathbf{n}$ (defined in Example 7) naturally gives rise to $n$ tabular logics $\models_{(\mathbf{n},\mathsf{e}_i)}$ of type $\nu_n$, for each $i \leq n$. These logics have a unique connective $q$ of arity $n+1$ and generalise classical propositional logic CL. As a matter of fact, the logic $\models_{(\mathbf{2},\mathsf{e}_2)}$ is nothing else than CL with a different choice of primitive connectives. For these reasons, for all $2 \leq n < \omega$, the logic $\models_{(\mathbf{n},\mathsf{e}_n)}$ of type $\nu_n$ will be called *classical logic of dimension* $n$ and denoted by $n$CL.

In this section we show:

(i) The complete symmetry of the truth values $\mathsf{e}_1, \ldots, \mathsf{e}_n$, supporting the idea that $n$CL is the right generalisation of classical logic from dimension 2 to dimension $n$.
(ii) The universality of $n$CL, by conservatively translating any $n$-valued tabular logic into it.
(iii) The existence of a terminating and confluent term rewriting system (TRS, for short) to test the validity of $\nu_n$-formulas by rewriting. By the universality of $n$CL, in order to check whether a $n$-valued tabular logic satisfies a formula $\phi$ it is enough to see whether the normal form of the translation $\phi^*$ is $\mathsf{e}_n$.

7.1. **Symmetry.** The choice of the designated value $\mathsf{e}_n$ is arbitrary, given the perfect symmetry among the truth values, as testified by Theorem 11 below.

Let $\mathbf{A}$ be an $n$BA and $\sigma, \rho$ be permutations of $1, \ldots, n$. For any $x \in A$, we recall that $x^\sigma$ stands for $q(x, \mathsf{e}_{\sigma 1}, \ldots, \mathsf{e}_{\sigma n})$, and we note that $(\mathsf{e}_i)^\sigma = \mathsf{e}_{\sigma i}$.

**Lemma 12.** (i) $(x^\sigma)^\rho = x^{\rho\circ\sigma}$;
(ii) $q(x, y_1, \ldots, y_n)^\sigma = q(x, y_1^\sigma, \ldots, y_n^\sigma)$;
(iii) $q(x^\sigma, y_1, \ldots, y_n) = q(x, y_{\sigma 1}, \ldots, y_{\sigma n})$.

*Proof.*

$$\begin{aligned}
(x^\sigma)^\rho &= q(q(x, \mathsf{e}_{\sigma 1}, \ldots, \mathsf{e}_{\sigma n}), \mathsf{e}_{\rho 1}, \ldots, \mathsf{e}_{\rho n}) \\
&= q(x, q(\mathsf{e}_{\sigma 1}, \mathsf{e}_{\rho 1}, \ldots, \mathsf{e}_{\rho n}), \ldots, q(\mathsf{e}_{\sigma n}, \mathsf{e}_{\rho 1}, \ldots, \mathsf{e}_{\rho n})) \quad \text{by B3} \\
&= q(x, \mathsf{e}_{\rho(\sigma 1)}, \ldots, \mathsf{e}_{\rho(\sigma n)}) \\
&= x^{\rho\circ\sigma}.
\end{aligned}$$

$$\begin{aligned}
q(x, y_1, \ldots, y_n)^\sigma &= q(q(x, y_1, \ldots, y_n), \mathsf{e}_{\sigma 1}, \ldots, \mathsf{e}_{\sigma n}) \\
&= q(x, y_1^\sigma, \ldots, y_n^\sigma) \quad \text{by B3}
\end{aligned}$$

$$\begin{aligned}
q(x^\sigma, y_1, \ldots, y_n) &= q(q(x, \mathsf{e}_{\sigma 1}, \ldots, \mathsf{e}_{\sigma n}), y_1, \ldots, y_n) \\
&= q(x, q(\mathsf{e}_{\sigma 1}, y_1, \ldots, y_n), \ldots, q(\mathsf{e}_{\sigma n}, y_1, \ldots, y_n)) \quad \text{by B3} \\
&= q(x, y_{\sigma 1}, \ldots, y_{\sigma n}).
\end{aligned}$$

□

Let $\sigma$ be a permutation of $1, \ldots, n$ and $\sigma^{-1}$ be the inverse permutation. We define $\mathbf{n}^\sigma$ to be the pure $n$BA with universe $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ such that

$$q^{\mathbf{n}^\sigma}(x, y_1, \ldots, y_n) = q^{\mathbf{n}}(x^{\sigma^{-1}}, y_1, \ldots, y_n).$$

Notice that $q^{\mathbf{n}}(x, y_1, \ldots, y_n) = q^{\mathbf{n}^\sigma}(x^\sigma, y_1, \ldots, y_n)$.

**Lemma 13.** *The map $x \mapsto x^\sigma$ defines an isomorphism from $\mathbf{n}$ onto $\mathbf{n}^\sigma$, whose inverse is the map $x \mapsto x^{\sigma^{-1}}$.*

*Proof.*
$$\begin{aligned} q^{\mathbf{n}}(x, y_1, \dots, y_n)^\sigma &= q^{\mathbf{n}}(x, y_1^\sigma, \dots, y_n^\sigma) \\ &= q^{\mathbf{n}}(x^{\sigma^{-1}\circ\sigma}, y_1^\sigma, \dots, y_n^\sigma) \\ &= q^{\mathbf{n}}((x^\sigma)^{\sigma^{-1}}, y_1^\sigma, \dots, y_n^\sigma) \\ &= q^{\mathbf{n}^\sigma}(x^\sigma, y_1^\sigma, \dots, y_n^\sigma). \end{aligned}$$
□

If $\phi$ is a $\nu_n$-formula and $\sigma$ is a permutation, we define $\phi^\sigma = q(\phi, \mathsf{e}_{\sigma 1}, \dots, \mathsf{e}_{\sigma n})$.

**Theorem 11.** *Let $\sigma$ be a permutation. Then the following conditions are equivalent:*

(i) $\Gamma \models_{(\mathbf{n}, \mathsf{e}_i)} \phi$;
(ii) $\Gamma^\sigma \models_{(\mathbf{n}, \mathsf{e}_{\sigma i})} \phi^\sigma$;
(iii) $\Gamma \models_{(\mathbf{n}^\sigma, \mathsf{e}_{\sigma i})} \phi$.

*Proof.* Let $\mathbf{T}_{\nu_n}$ be the absolutely free algebra of the $\nu_n$-formulas over a countable set of variables, and let $g : \mathbf{T}_{\nu_n} \to \mathbf{n}$ be a homomorphism.

(i) $\Leftrightarrow$ (ii) holds because $g(\psi) = \mathsf{e}_i$ iff $g(\psi^\sigma) = \mathsf{e}_{\sigma i}$, for every homomorphism $g$.

(i) $\Leftrightarrow$ (iii): By Lemma 13 $h : \mathbf{T}_{\nu_n} \to \mathbf{n}^\sigma$ is a homomorphism iff $(-)^{\sigma^{-1}} \circ h : \mathbf{T}_{\nu_n} \to \mathbf{n}$ is a homomorphism. Then $h(\psi) = \mathsf{e}_{\sigma i}$ iff $h(\psi)^{\sigma^{-1}} = \mathsf{e}_i$. □

**Example 11.** *Let $\phi, \psi$ be two formulas of classical logic CL and let $\sigma$ be the permutation such that $\sigma 1 = 2$ and $\sigma 2 = 1$. Recalling that $\mathsf{t} = \mathsf{e}_2$ and $\mathsf{f} = \mathsf{e}_1$, we have $\psi \models_{(\mathbf{2},1)} \phi \Leftrightarrow \psi^\sigma \models_{(\mathbf{2},0)} \phi^\sigma \Leftrightarrow \neg\psi \models_{(\mathbf{2},0)} \neg\phi$.*

7.2. **Universality.** The logic $n$CL is universal in the sense that any $n$-valued tabular logic with a single designated value admits a faithful translation into $n$CL.

Let L be a tabular logic $\models_{(\mathbf{A},\mathsf{t})}$ of type $\tau$ with a single designated value $\mathsf{t}$ such that $|A| = n$. In what follows, we identify $A$ with $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$ in such a way that the designated element $\mathsf{t}$ is equal to $\mathsf{e}_n$.

We now translate the formulas of $L$ into formulas of $n$CL. We start with the logical connectives. If $f$ is a $k$-ary connective of type $\tau$, then $f^{\mathbf{A}} : A^k \to A$ is a $k$-ary operation on $\mathbf{A}$. As the algebra $\mathbf{A}$ and the $n$BA $\mathbf{n}$ have the same universe, then $f^{\mathbf{A}}$ can be considered as a function from $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}^k$ to $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$. By Lemma 5 the function $f^{\mathbf{A}}$ is term definable through a hnf $f^\circ(x_1, \dots, x_k)$ of type $\nu_n$. The translation of the $\tau$-formulas into $\nu_n$-formulas is defined by induction as follows:

$$x^* = x; \quad \mathsf{e}_i^* = \mathsf{e}_i; \quad f(\phi_1, \dots, \phi_k)^* = f^\circ(\phi_1^*/x_1, \dots, \phi_k^*/x_k).$$

The formula $f(\phi_1, \dots, \phi_k)^*$ is not in general a hnf, because the substitution $\phi_i^*/x_i$ may occur into a head occurrence of the variable $x_i$.

The next theorem shows that the translation is sound and complete.

**Theorem 12.** *Let L be a tabular logic $\models_{(\mathbf{A},\mathsf{t})}$ of type $\tau$ with a single designated value $\mathsf{t}$ such that $|A| = n$. Then $n$CL is a conservative expansion of L and, for $\tau$-formulas $\Gamma \cup \{\phi\}$, we have:*

$$\Gamma \models_{(\mathbf{A},\mathsf{t})} \phi \;\Leftrightarrow\; \Gamma^* \models_{(\mathbf{n},\mathsf{e}_n)} \phi^*.$$

**Example 12.** *The translation of the connectives of CL is as follows, where $0 = \mathsf{e}_1$ and $1 = \mathsf{e}_2$:*

$$\vee^\circ = q(x_1, x_2, 1) \qquad \wedge^\circ = q(x_1, 0, x_2); \qquad \neg^\circ = q(x_1, 1, 0).$$

**Example 13.** *The translation of the connectives of Gödel Logic $\mathcal{G}_3$, Lukasiewicz Logic $\L_3$, and Post Logic $\mathcal{P}_3$, defined in Example 1, is as follows:*

- *$\mathcal{G}_3$, $\L_3$, $\mathcal{P}_3$:* $\vee^\circ = q(x, y, q(y, \mathsf{e}_2, \mathsf{e}_2, \mathsf{e}_3), \mathsf{e}_3) \qquad \wedge^\circ = q(x, \mathsf{e}_1, q(y, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_2), y)$

- $\mathcal{G}_3$: $\neg^\circ = q(x, \mathsf{e}_3, \mathsf{e}_1, \mathsf{e}_1)$; $\quad \mathcal{L}_3$: $\neg^\circ = q(x, \mathsf{e}_3, \mathsf{e}_2, \mathsf{e}_1)$; $\quad \mathcal{P}_3$: $\neg^\circ = q(x, \mathsf{e}_3, \mathsf{e}_1, \mathsf{e}_2)$;
- $\mathcal{G}_3$: $\to^\circ = q(x, \mathsf{e}_3, q(y, \mathsf{e}_1, \mathsf{e}_3, \mathsf{e}_3), y)$; $\quad \mathcal{L}_3$: $\to^\circ = q(x, \mathsf{e}_3, q(y, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_3), y)$.

7.3. **Rewriting.** In this section we show how to turn the equations axiomatising $n$BA into rewriting rules. We introduce two terminating and confluent TRSs $\rightarrowtail_{\mathrm{hnf}}$ and $\rightarrowtail_{\mathrm{full}}$ on $\nu_n$-formulas. We will show that

$$\models_{(\mathbf{n}, \mathsf{e}_n)} \phi \quad \text{iff} \quad \phi \rightarrowtail^*_{\mathrm{hnf}} \mathrm{hnf}(\phi) \rightarrowtail^*_{\mathrm{full}} \mathsf{e}_n,$$

where $\mathrm{hnf}(\phi)$ is the canonical head normal form logically equivalent to $\phi$ (see Definition 11 and Lemma 2).

The rewriting rules in this section are very tightly related to the equivalence transformation rules of multi-valued decision diagrams (see [11, 19]). The TRS $\rightarrowtail_{\mathrm{full}}$ is a direct generalisation to the multiple case of Zantema and van de Pol binary rewriting system described in [28], where the authors are seemingly unaware that their axiomatisation captures exactly binary decomposition operators.

The section is organised as follows: in Section 7.3.1 we prove that the TRS $\rightarrowtail_{\mathrm{hnf}}$ is confluent and terminating. Then in Section 7.3.2 we describe the relationship between multiple-valued decision diagrams and $n$BAs. We conclude the article with Section 7.3.3, where the TRS $\rightarrowtail_{\mathrm{full}}$ is presented, and its termination and confluence are stated.

7.3.1. *The hnf of a formula.* In this section we define a confluent and terminating rewriting system to get the canonical hnf of a formula.

We consider the variety $\mathcal{H}$ of $\nu_n$-algebras axiomatised by the following identities:

(A1) $q(\mathsf{e}_i, x_1, \ldots, x_n) = x_i$;
(A2) $q(q(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = q(x, q(y_1, z_1, \ldots, z_n), \ldots, q(y_n, z_1, \ldots, z_n))$.

Notice that (A1) is the axiom defining $n$DAs and (A2) is an instance of axiom (B3) defining $n$-central elements.

We define an algebra $\mathbf{H}$ of type $\nu_n$, having the set of hnfs as universe. The operation $q^{\mathbf{H}}$ is defined by induction over the complexity of its first hnf argument.

For every hnfs $\bar{\psi} = \psi_1, \ldots, \psi_n$:

$$q^{\mathbf{H}}(\mathsf{e}_i, \bar{\psi}) = \psi_i$$
$$q^{\mathbf{H}}(x, \bar{\psi}) = q(x, \bar{\psi}) \quad (x \in \mathrm{Var})$$
$$q^{\mathbf{H}}(q(x, u_1, \ldots, u_n), \bar{\psi}) = q(x, q^{\mathbf{H}}(u_1, \bar{\psi}), \ldots, q^{\mathbf{H}}(u_n, \bar{\psi})).$$

A routine calculation shows that $\mathbf{H}$ is isomorphic to the free algebra $\mathbf{F}_{\mathcal{H}}$ over a countable set Var of generators.

We turn the identities axiomatising $\mathcal{H}$ into rewriting rules.

**Definition 16.** *The rewriting rules $\rightarrowtail_{\mathrm{hnf}}$ are:*

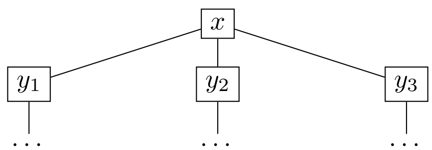
**($h_0$):** $q(\mathsf{e}_i, x_1, \ldots, x_n) \rightarrowtail_{\mathrm{hnf}} x_i$
**($h_1$):** $q(q(x, y_1 \ldots, y_n), z_1, \ldots, z_n) \rightarrowtail_{\mathrm{hnf}} q(x, q(y_1, z_1, \ldots, z_n), \ldots, q(y_n, z_1, \ldots, z_n))$.

**Theorem 13.** *The rewriting system $\rightarrowtail_{\mathrm{hnf}}$ is terminating and confluent.*

*Proof.* The left linear system $\rightarrowtail_{\mathrm{hnf}}$ is confluent because all critical pairs are converging. Termination is obtained by applying the subterm criterion to the dependency pairs of $\rightarrowtail_{\mathrm{hnf}}$ (see [12, Section 2]). We have the following dependency pairs of rule ($h_1$) $l \rightarrowtail_{\mathrm{hnf}} r$: $q^{\#}(q(x, \bar{y}), \bar{z}) \rightarrowtail q^{\#}(y_i, \bar{z})$, since $q(y_i, \bar{z})$ is not a subterm of $l$, and $q^{\#}(q(x, \bar{y}), \bar{z}) \rightarrowtail q(x, q(y_1, \bar{z}), \ldots, q(y_n, \bar{z}))$, since $r$ is not a subterm of $l$. For the subterm criterion, we apply the simple projection in the first argument. This proof of termination of the TRS $\rightarrowtail_{\mathrm{hnf}}$ is automatised by the tool $\mathsf{T_TT_2}$ (see [26]). $\square$

The normal forms of $\rightarrowtail_{\mathrm{hnf}}$ are the hnfs; we denote by $\mathrm{hnf}(\phi)$ the normal form of $\phi$.

7.3.2. *$n$CL and decision diagrams.* The theory of $n$BAs, and of the related logics $n$CL, is strictly related to binary decision diagrams (BDD) and multivalued decision diagrams (MDD). A decision diagram is an acyclic oriented graph that can be unfolded as a tree. Each branch node represents a choice between a number of alternatives and each leaf node represents a decision. As the operation $q$ in the $n$BA $\mathbf{n}$ is an $n$-arguments choice operation, a decision tree with branching factor at most $n$, whose nodes are labelled only by variables, can be codified as a head normal form in the type of pure $n$BAs. As a matter of fact, a logical variable $x$ labelling a node of a decision diagram is an operator, whose arity is the branching factor of the node. For example, the variable $x$ in the diagram $D$ below is a ternary operator:




This diagram is naturally represented by the head normal form:

$$\phi_D = q(x, q(y_1, \dots), q(y_2, \dots), q(y_3, \dots)).$$

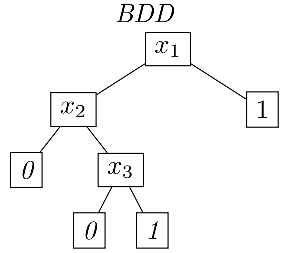
In general, a branch of an $n$-branching decision tree $D$ in $k$ variables corresponds to a homomorphism from the term algebra $\mathbf{T}_{\nu_n}(y_1, \dots, y_k)$ into $\mathbf{n}$. In an arbitrary $n$BA an $n$-branching variable $x$ of a decision diagram becomes the decomposition operator $q(x, -_1, \dots, -_n)$.

It is remarkable that several transformations on decision diagrams found in literature [11, 19] are instances of $n$BA axiomatisation.

The next example explains the relationship among CL, 2BA and BDDs.

**Example 14.** *By Example 12 the formula $\phi = x_1 \vee (x_2 \wedge x_3)$ is translated into 2CL as follows: $\phi^* = q(x_1, q(x_2, 0, x_3), 1)$ (where $0 = \mathsf{e}_1$ and $1 = \mathsf{e}_2$).*

*CL* *2CL* *BDD*



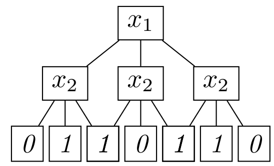

$\phi = x_1 \vee (x_2 \wedge x_3)$ $\phi^* = q(x_1, q(x_2, 0, x_3), 1)$

*Remark that the formula $\phi^*$ in the above example is a hnf, and this allows to associate a BDD to it. In general, the translation of a formula is not in head normal form, as for instance $((x_1 \vee x_2) \wedge x_3)^* = q(q(x_1, x_2, 1), 0, x_3)$. Applying the TRS $\rightarrowtail_{\mathrm{hnf}}$ to the formula $q(q(x_1, x_2, 1), 0, x_3)$ we get an equivalent head normal form. Thus, we may associate univocally a decision diagram to any formula.*

In the next example, we consider a non-binary case: the "all-different" constraint for ternary variables. To keep the example reasonably small, we consider the simple case of two variables.

**Example 15.** *Let $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3$ be the truth values. We define $1 = \mathsf{e}_3$ and $0 = \mathsf{e}_1$.*

*2CL* *BDD*




$$\phi^* = q(x_1, q(x_2, 0, 1, 1), q(x_2, 1, 0, 1), q(x_2, 1, 1, 0))$$

7.3.3. *The normal form of a formula.* We introduce a second TRS, which, restricted to hnfs, is terminating and confluent. First we define the variety $\mathcal{W}$ axiomatised over $\mathcal{H}$ (see Section 7.3.1) by the following identities:

(A3) $q(x, y, \dots, y) = y$;
(A4) $q(x, \mathsf{e}_1, \dots, \mathsf{e}_n) = x$;
(A5$^i$) $q(x, y_1, \dots, y_{i-1}, q(x, z_1, \dots, z_n), y_{i+1}, \dots, y_n) = q(x, y_1, \dots, y_{i-1}, z_i, y_{i+1}, \dots, y_n)$;
(A6) $q(x, q(y, y_1^1, \dots, y_n^1), \dots, q(y, y_1^n, \dots, y_n^n)) = q(y, q(x, y_1^1, \dots, y_1^n), \dots, q(x, y_n^1, \dots, y_n^n))$.

**Lemma 14.** $\mathcal{W} = n\mathrm{BA}$.

*Proof.* $(B3) : q(q(c, x_1, \dots, x_n), q(c, \bar{y}), \dots, q(c, \bar{z})) =_{\mathrm{A2}}$
$q(c, q(x_1, q(c, \bar{y}), \dots, q(c, \bar{z})), \dots, q(x_n, q(c, \bar{y}), \dots, q(c, \bar{z}))) =_{\mathrm{A6}}$
$q(c, q(c, q(x_1 y_1 \dots z_1), \dots, q(x_1 y_n \dots z_n)), \dots, q(c, q(x_n y_1 \dots z_1), \dots, q(x_n y_n \dots z_n)) =_{\mathrm{A5^i}}$
$q(c, q(x_1, y_1, \dots, z_1), \dots, q(x_n, y_n, \dots, z_n))$. □

By Lemma 14 the free algebra $\mathbf{F}_{n\mathrm{BA}}$ over a countable set Var of generators is isomorphic to the quotient $\mathbf{F}_{\mathcal{H}}/\theta$, where $\theta$ is the fully invariant congruence generated by the axioms (A3)-(A6). The characterisation of $\mathbf{F}_{\mathcal{H}}$ given in Section 7.3 justifies the restriction to hnfs of the TRS defined below.

For all $\nu_n$-formulas $\phi$ of $n$CL, each subterm of hnf($\phi$) of the form $q(x, y_1, \dots, y_n)$ is such that $x$ is always a variable. It follows that the head occurrences of variables in hnf($\phi$) behave as constants. We consider a total order $<$ on the set Var of variables: $x_1 < x_2 < x_3 < \dots$.

**Definition 17.** *The following are the rewriting rules $\rightarrowtail_{\mathrm{full}}$ acting on hnfs, where $x, x'$ ranges over variables and $y, y_i, z_j, \bar{u}$ over arbitrary hnfs:*

**($r_2$):** $q(x, y, \dots, y) \rightarrowtail y$;
**($r_3$):** $q(x, \mathsf{e}_1, \dots, \mathsf{e}_n) \rightarrowtail x$;
**($r_4^i$):** $q(x, y_1, \dots, y_{i-1}, x, y_{i+1}, \dots, y_n) \rightarrowtail q(x, y_1, \dots, y_{i-1}, \mathsf{e}_i, y_{i+1}, \dots, y_n)$;
**($r_5^i$):** $q(x, y_1, \dots, y_{i-1}, q(x, z_1, \dots, z_n), y_{i+1}, \dots, y_n) \rightarrowtail q(x, y_1, \dots, y_{i-1}, z_i, y_{i+1}, \dots, y_n)$;
**($r_6^i$):** *If* $x' < x$ *then* $q(x, y_1, \dots, y_{i-1}, q(x', z_1, \dots, z_n), y_{i+1}, \dots, y_n) \rightarrowtail$
$q(x', q(x, y_1, \dots, y_{i-1}, z_1, y_{i+1}, \dots, y_n), \dots, q(x, y_1, \dots, y_{i-1}, z_n, y_{i+1}, \dots, y_n))$
**($r_7^i$):** *If* $x' < x$ *then* $q(x, y_1, \dots, y_{i-1}, x', y_{i+1}, \dots, y_n) \rightarrowtail$
$q(x', q(x, y_1, \dots, y_{i-1}, \mathsf{e}_1, y_{i+1}, \dots, y_n), \dots, q(x, y_1, \dots, y_{i-1}, \mathsf{e}_n, y_{i+1}, \dots, y_n))$

**Theorem 14.** $\rightarrowtail_{\mathrm{full}}$ *restricted to hnfs is terminating and confluent.*

*Proof.* The proof is an easy generalisation of the case $n = 2$ that can be found in [28, 21]. For the sake of completeness, the proof is given in the appendix. □

**Corollary 4.** *A given $n$-valued tabular logic satisfies $\models_{(\mathbf{A},\mathrm{t})} \phi$ if and only if $\phi^* \rightarrowtail^* \mathsf{e}_n$.*

## 8. Conclusion

The $n$-dimensional Boolean algebras introduced in this paper provide a generalisation of Boolean algebras, since many remarkable properties of the variety of BAs are inherited by $n$BAs. We have shown some of these properties, for instance the fact that the $n$BA $\mathbf{n}$ plays a role analogous to the Boolean algebra **2** of truth values, and left other aspects of the theory for further work; for instance, the analogous of ultrafilters and congruences, the relationship between $n$BAs and skew Boolean algebras (see Leech [13]).

We have introduced the propositional logic $n$CL, and a confluent and terminating rewriting system that allows to decide the validity of $n$CL formulas. Since all tabular logics may be faithfully embedded into $n$CL, our rewriting system may be used to check the validity in any tabular logic.

Interestingly, the framework developed here provides new foundations and offers new perspectives to the field of multivalued logics and decision diagrams: in multivalued decision diagram, the MV-CASE primitive generalises the if-then-else (ITE) commonly used in binary decision diagrams. That is exactly the operation $q$ of the $n$BAs. Lot of work is devoted to the optimisation of multivalued decision diagram [19]; the sound and complete axiomatisation of the MV-CASE given here can be a valuable tool in that field.

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## 9. Appendix

In this appendix we prove Theorem 14. Let $\Sigma$ be the type given by: $\{c_i\}_{i\in N}$ and $\mathtt{e}_1,...,\mathtt{e}_n$, nullary function symbols, and $q$, $(n+1)$-ary function symbol. We write $c_i < c_j$ if $i < j$. The constants $c_i$ represent the elements of Var.

**Definition 18.** *The lexicographic path ordering on hnfs $<_{lpo}$ is defined by $t <_{lpo} u$ if:*

- $(b_1)$ *$\exists i, j$ such that $t = \mathtt{e}_j$ and $u = c_i$.*
- $(b_2)$ *$\exists i < j$ such that $t = c_i$ and $u = c_j$.*
- $(s_1)$ *$u = q(u_0, u_1, ..., u_n)$ and $\exists i \in \{0, ..., n\}$ such that $t \leq_{lpo} u_i$, where $t \leq_{lpo} u_i$ stands for $t <_{lpo} u_i$ or $t = u_i$.*
- $(s_2)$ *$t = q(t_0, t_1, ..., t_n), u = q(u_0, u_1, ..., u_n)$ and $\exists i \in \{0, ..., n\}$ such that $\forall j \in \{0, ..., i-1\}$ $t_j = u_j$, $t_i <_{lpo} u_i$ and $\forall j \in \{i+1, ..., n\}$ $t_j <_{lpo} u$.*

**Lemma 15.** *For each rewriting rule $t \rightarrowtail u$ of $\mathcal{R}_{\mathrm{full}}$ we have $t >_{lpo} u$.*

*Proof.*

- $(r_2)$: $q(c_j, y, \dots, y) >_{lpo} y$ by $(s_1)$.
- $(r_3)$: $q(c_j, \mathtt{e}_1, \dots, \mathtt{e}_n) >_{lpo} c_j$ by $(s_1)$.
- $(r_4^i)$: $q(c_j, y_1, \dots, y_{i-1}, c_j, y_{i+1}, \dots, y_n) >_{lpo} q(c_j, y_1, \dots, y_{i-1}, \mathtt{e}_i, y_{i+1}, \dots, y_n)$ by $(s_2)$ and $(b_1)$.
- $(r_5^i)$: $q(c_j, y_1, \dots, y_{i-1}, q(c_j, z_1, \dots, z_n), y_{i+1}, \dots, y_n) >_{lpo} q(c_j, y_1, \dots, y_{i-1}, z_i, y_{i+1}, \dots, y_n)$ by $(s_2)$ and $(s_1)$.
- $(r_6^i)$: if $c_k < c_j$ then $q(c_j, y_1, \dots, y_{i-1}, q(c_k, z_1, \dots, z_n), y_{i+1}, \dots, y_n) >_{lpo}$ $q(c_k, q(c_j, y_1, \dots, y_{i-1}, z_1, y_{i+1}, \dots, y_n), \dots, q(c_j, y_1, \dots, y_{i-1}, z_n, y_{i+1}, \dots, y_n))$ by $(s_2)$, $(b_2)$ and $(s_1)$.
- $(r_7^i)$: if $c_k < c_j$ then $q(c_j, y_1, \dots, y_{i-1}, c_k, y_{i+1}, \dots, y_n) >_{lpo}$ $q(c_k, q(c_j, y_1, \dots, y_{i-1}, \mathtt{e}_1, y_{i+1}, \dots, y_n), \dots, q(c_j, y_1, \dots, y_{i-1}, \mathtt{e}_n, y_{i+1}, \dots, y_n))$ by $(s_2)$ and $(b_1)$.

□

Since the partial order on $\Sigma$ given by $(b_1)$ and $(b_2)$ in Definition 18 is well founded, the corresponding recursive path ordering $<_{lpo}$ on hnfs is well founded, too. Hence:

**Theorem 15.** *The rewriting system $\mathcal{R}_{\mathrm{full}}$ is terminating.*

The confluence of $\mathcal{R}_{\mathrm{full}}$ is proved by showing that two $\mathcal{R}_{\mathrm{full}}$ normal forms that are logically equivalent are actually equal.

**Definition 19.** *Let $C = \{c_n\}_{n\in\mathbb{N}}$. Given an* environment $\rho : C \to \mathbf{n}$ *let us define the* interpretation of the term $t$ with respect to $\rho$, *written* $[\![t]\!]_\rho \in \mathbf{n}$, *as follows:*

- $[\![\mathsf{e}_i]\!]_\rho = \mathsf{e}_i$
- $[\![c_i]\!]_\rho = \rho(c_i)$
- $[\![q(t, u_1, ..., u_n)]\!]_\rho = [\![u_i]\!]_\rho$ *if* $[\![t]\!]_\rho = \mathsf{e}_i$.

*The terms $t$ and $u$ are* logically equivalent, *written $t \simeq u$, if for all $\rho$, $[\![t]\!]_\rho = [\![u]\!]_\rho$.*

The following fact is trivial:

**Fact 1.** *If $c_i$ does not occur in the term $t$, and the environments $\rho, \rho'$ are such that for all $j \neq i$ $\rho(c_j) = \rho'(c_j)$, then $[\![t]\!]_\rho = [\![t]\!]_{\rho'}$.*

As a matter of notation, for $i \in \mathbb{N}$, $1 \leq k \leq n$ and an environment $\rho$ let $\rho_{i\leftarrow k}$ be the environment defined by

$$\rho_{i\leftarrow k}(c_j) = \begin{cases} \rho(c_j) & \text{if } j \neq i \\ \mathsf{e}_k & \text{otherwise} \end{cases}$$

**Definition 20.** *The* size *of $\Sigma$-terms is defined by*

- $\#(c_i) = \#(\mathsf{e}_j) = 0$,
- $\#(q(u_0, u_1, ..., u_n)) = 1 + \#(u_0) + ... + \#(u_n)$.

**Lemma 16.** *If $t = q(c_i, t_1, ..., t_n)$ is a $\mathcal{R}_{\text{full}}$ normal form, and if $c_j$ occurs in $t_k$, $j \in \mathbb{N}$ and $1 \leq k \leq n$, then $c_i < c_j$.*

*Proof.* By induction on $t_k$. If $t_k = c_j$, then $c_i < c_j$ since if $c_i = c_j$ then the rule $r_4^k$ applies to $t$, and if $c_j < c_i$ then the rule $r_7^k$ applies to $t$. Otherwise, $t_k = q(c_l, u_1, ..., u_n)$. By induction hypothesis we know that $c_l \leq c_j$. We also know that $c_l \neq c_i$, otherwise rule $r_5^k$ applies to $t$, and that $c_l \not< c_i$, otherwise rule $r_6^k$ applies to $t$. Hence $c_i < c_l$ and we are done. □

**Lemma 17.** *Let $t, u$ be $\mathcal{R}_{\text{full}}$ normal forms:*

- *If $t \simeq u$ then $t = u$.*
- *If $t = q(c_i, t_1, ..., t_n)$ (resp. $u = q(c_j, u_1, ..., u_n)$) then there exist $1 \leq k, l \leq n$ such that $t_k \not\simeq t_l$ (resp. $u_k \not\simeq u_l$ ) .*

*Proof.* Let us call $S(t, u)$ and $T(t, u)$ the two statement of the lemma, whose proof is by mutual induction on $\#(t) + \#(u)$.
If $\#(t) + \#(u) = 0$ then:

- $T(t, u)$: trivial.
- $S(t, u)$: $t$ and $u$ are either $c_i$ or $\mathsf{e}_j$ for some $i, j$; it is easy to see that if $t \neq u$ then $t \not\simeq u$.

If $\#(t) = 0$ and $\#(u) > 0$, say $u = q(c_i, u_1, ..., u_n)$:

- $T(t, u)$: if for all $1 \leq i, j \leq n$ we had $u_i \simeq u_j$, then by the induction hypothesis $S(u_i, u_j)$ we would get $u_i = u_j$, and hence $u$ would be a $(r_2)$-redex.
- $S(t, u)$: We reason by cases on $t$: if $t = \mathsf{e}_j$, then $t \simeq u$ iff for all $i$ $u_i \simeq \mathsf{e}_j$. By induction hypothesis $S(\mathsf{e}_j, u_i)$, we have that for all $i$ $u_i = \mathsf{e}_j$, and hence $u$ is a $(r_2)$-redex, a contradiction. If $t = c_i$, then $t \simeq u$ iff for all $i$ $u_i \simeq \mathsf{e}_i$. By the induction hypothesis $S(\mathsf{e}_i, u_i)$, , we have for all $i$ $u_i = \mathsf{e}_i$, and hence $u$ is a $(r_3)$-redex, a contradiction. If $t = c_j$ for some $j \neq i$, then let $k, l, \rho$ given by the induction hypothesis $T(t, u)$, such that $[\![u_k]\!]_\rho \neq [\![u_l]\!]_\rho$. Let us suppose w.l.o.g. that $[\![u_k]\!]_\rho \neq \rho(c_j)$, otherwise we pick $l$ instead of $k$, and let $\rho' = \rho_{i\leftarrow k}$. Using Lemma 16 and Fact 1 we get:

$$[\![u]\!]_{\rho'} = [\![u_k]\!]_{\rho'} = [\![u_k]\!]_\rho \neq \rho(c_j) = \rho'(c_j) = [\![t]\!]_{\rho'}$$

  a contradiction, since $u \simeq t$.

If $\#(t) > 0$ and $\#(u) = 0$, we reason as above.
If $\#(t) > 0$ and $\#(u) > 0$, say $t = q(c_i, t_1, ..., t_n)$ and $u = q(c_j, u_1, ..., u_n)$:

- $T(t, u)$: as above, if for all $1 \leq i, j \leq n$ we had $u_i \simeq u_j$, then by induction hypothesis $S(u_i, u_j)$ we would get $u_i = u_j$, and hence $u$ would be a $(r_2)$-redex. Similarly for $t$.
- $S(t, u)$ we proceed by case analysis: if $i = j$, we have that $t \simeq u$ iff for all $i$, $t_i \simeq u_i$. Then the induction hypothesis $S(t_i, u_i)$ gives $t_i = u_i$, and hence $t = u$. If $i \neq j$, then w.l.o.g. let us suppose that $c_i < c_j$. By the induction hypothesis $T(t, u)$, let $[\![t_k]\!]_\rho \neq [\![t_l]\!]_\rho$, and let us suppose, again w.l.o.g., that $[\![u]\!]_\rho \neq [\![t_k]\!]_\rho$ (otherwise we pick $l$ instead of $k$). By using Lemma 16 and Fact 1 we get:

$$[\![t]\!]_{\rho_{i \leftarrow k}} = [\![t_k]\!]_{\rho_{i \leftarrow k}} = [\![t_k]\!]_\rho \neq [\![u]\!]_\rho = [\![u]\!]_{\rho_{i \leftarrow k}}$$

a contradiction, since $u \simeq t$.

□

**Lemma 18.** *If $t \rightarrowtail u$ in $\mathcal{R}_{\text{full}}$, then $t \simeq u$.*

**Theorem 16.** *The rewriting system $\mathcal{R}_{\text{full}}$ is confluent.*

*Proof.* If $t \rightarrowtail^* t_i$ for $i = 1, 2$, let $t'_i$ be the $\mathcal{R}_{\text{full}}$ normal form of $t_i$, that exists by Theorem 15. By Lemma 18 $t'_1 \simeq t \simeq t'_2$, and we conclude by Lemma 17 that $t'_1 = t'_2$. □

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