POLICY EFFECTIVENESS IN SPATIAL RESOURCE WARS: A TWO-REGION MODEL

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Abstract. We develop a spatial resource model in continuous time in which two agents strategically exploit a mobile resource in a two-region setup.

To counteract the overexploitation of the resource (the *tragedy of commons*) that occurs when players are free to choose where to harvest, the regulator can establish a series of spatially structured policies. We compare the equilibria in the case of a common resource with those that emerge when the regulator either creates a natural reserve, or assigns Territorial User Rights to the players.

We show that, when technological and preference parameters dictate a low harvesting effort, the policies are ineffective in promoting the conservation of the resource and, in addition, they lead to a lower payoff for at least one of the players. Conversely, in a context of higher harvesting effort, the intervention can help to safeguard the resource, preventing extinction while also improving the welfare of both players.

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JEL Classification: Q28, C72, Q23, C61, R12.

1. Introduction

Traditional management tools have often been unsuccessful in preventing the rapid decline of natural resource stocks. As a consequence, over the last few decades the management of renewable and exhaustible resources has been increasingly enforced via property rights, in particular via spatial rights. Practiced over centuries in some parts of the world (Japan, for example), resource management via spatial rights has now spread worldwide (see Quynh, Schilizzi, Hailu and Iftekhar, 2017, for fisheries), and is usually introduced to address the lack of well-defined property rights of the commons. However, only a few of the resources involved are completely immobile. Fish stocks, to take an obvious example, are spatially distributed

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and in many cases move across different locations. Similarly, stocks of air or water pollutants are rarely stationary at the emission point, but diffuse in space. Even water reservoirs, and some exhaustible resources such as oil deposits, have spatial dynamics. Hence the theoretical literature has warned from the start that if a *spatially distributed* resource moves across different locations in a fully connected network, then Territorial User Rights (TURF in the case of fisheries), which assign units of space to single agents, cannot be expected to be very effective at solving the overexploitation problem that tends to arise under common property (e.g, Janmaat, 2005; White and Costello, 2011; Kaffine and Costello, 2011; Costello, Quérou, and Tomini, 2015; Quérou, Costello, and Tomini 2018). For all that matters, the spatial externality generated by the migration of the stocks allows each agent with access to the network at a single node to actually access the whole resource. So ill-defined property rights persist after the introduction of Territorial User Rights.

In recent years, the fields of growth theory and environmental and resource economics have developed tools to face the challenging task of modeling the economic forces that shape the dynamics of extraction of moving spatially distributed stocks (see e.g., Smith, Sanchirico and Wilen, 2009; Xepapadeas, 2010; and Brock, Xepapadeas and Yannacopoulos, 2014, for surveys). In other works, optimal harvesting is studied for of an immobile spatially distributed resource, see for instance Behringer and Upmann (2014). Dynamic strategic interaction, however, is largely absent from the studies that have introduced spatial-dynamic processes in growth or resource models. In these works, the analysis generally proceeds either on the assumption that rent dissipates instantaneously (e.g., Sanchirico and Wilen, 1999), or on the assumption that the planner either controls the entire environment (e.g, Boucekkine, Camacho and Fabbri, 2013), or takes the spatially distributed stock path as given (e.g., Janmaat, 2005, Santambrogio, Xepapadeas, Yannacoupolos, 2017). Clearly, these assumptions are not well suited to the analysis of the spatial externalities that arise when the resource is a moving spatial distributed stock but access is restricted to a small number of extractors. As a consequence, there are still very few studies that contain analytically or numerically tractable dynamic games (Bhat and Huffaker, 2007; Kaffine and Costello, 2011; Herrera, Moeller and Neubert, 2016, Costello, Nkuiya and Quérou, 2019, Quérou, Costello and Tomini, 2018, de Frutos and Martin Herran, 2019).

On the other hand, stationary Markov perfect Nash equilibria in models with a single common-property resource have been studied under different hypotheses in the literature (see e.g., Levhari and Mirman, 1980, Clemhout and Wan, 1985, Negri, 1989, Tornell and Velasco, 1992, Dockner and Sorger, 1996, Sorger, 1998, Tornell and Lane, 1999, Rowat and Dutta, 2007, Strulik, 2012a, 2012b, Mitra and Sorger, 2014, 2015, Dasgupta, Mitra and Sorger, 2019 and, for a survey of the literature, Long, 2011, 2016). In the typical setting a homogeneous stock, whose growth function is known, is harvested by a finite number or mass of identical agents who reap utility from consuming the resource. Since analysis of the Markov perfect Nash equilibria has turned out to be difficult, straightforward results have been obtained only for special growth and utility functions (see e.g., Dockner, Jorgensen, Van Long and Sorger, 2000, section 12.1 for the case of an exhaustible resource with a isoelastic instantaneous utility function). Although there are a few exceptions, the usual conclusion in this literature is that non-cooperation leads to overexploitation of the resource (the so-called "tragedy of the commons").

In this work we develop a simplified framework to study "resource wars" with spatially distributed stocks. Our aim is to provide an analytically tractable model that generalizes some of the results obtained in the literature that studied Markov perfect equilibria in differential games with a homogeneous stock, and to highlight how difficult it is to design efficient systems for the management of resources based on spatial property rights, if the spatial externalities stemming from the movements of the stocks are not completely internalized. We compare the behaviors of agents in an initial *common property case*, in which they can decide both where and how much to harvest, with their choices in policy-constrained cases, in which the regulator can establish a natural reserve or assign a harvesting location to each agent. We show that implementing these policies can only be effective when the agents choose a high harvesting effort.

To have an analytically solvable model, some simplifications are made. First, we have chosen to study a two-region, two-player case. Second, as is often assumed in the literature, we suppose that the stock diffuses at a constant rate from the higher density to the lower density location. Third, special growth and utility functions are used since, as it is well known, not even mere existence results for Nash equilibria can be obtained in a general framework. In particular, as we look for linear Markov equilibria, tight restrictions must be imposed on the primitives of the model: we use throughout the paper the family of isoelastic utility functions and linear (re)production functions (see Gaudet and Lohoues, 2008, for the analysis of the conditions that guarantee the existence of a linear Markovian equilibrium in the scalar common pool case).

For the case in which the preferences of the agents and the technology dictate low harvesting effort, the existence of a Markov perfect Nash equilibrium is proved and explicitly characterized in three scenarios: (a) the initial *common property case*, where each agent can decide, at any times, in which regions to harvest in and by how much; (b) the *reserve case*, where the regulator forbids agents from harvesting in one of the two regions; (c) the *TURF Case* (where TURF stands for Territorial User Rights for Fisheries, as fishery is the straightforward application of the model), where each player can only harvest in an exclusive region. In each situation we characterize the optimal response function of the players, the resource stock evolution (in particular its reproduction rate at equilibrium) and the utility of the players.

It turns out that, in case of low harvesting effort, the mentioned spatial property rights cannot improve the growth rate of the resource and in particular they cannot prevent its depletion in case the implicit rates of growth is positive but small. In addition, their effect on the utilities of the agents is never positive and the policies strictly worsen the utility of at least one of the players.

The analysis of the results allows on the one hand to show (see Subsection 4.3) that if the elasticity of intertemporal substitution is sufficiently high (higher than 2 with two player, as in our basic model), a *voracity effect* (similar to that described by Tornell and Lane (1999)) arises in our spatial context, and on the other hand to identify what kinds of "technological" shocks generate it. As expected, if voracity prevails, then an increase in any of the local intrinsic growth rates of the resource reduces growth. Notably, however, it turns out that a reduction in the spatial mobility of the resource has the same effect.

Things change sharply when a policy induces agents to choose maximal effort (Section 5). Indeed, in the high-intensity harvesting case, the territorial policies we mentioned, and in particular the creation of a reserve, can lead to an effective reduction of the overexploitation and have a consequent positive impact on the rate of reproduction of the resource. The policies can prevent the asymptotic depletion of the resource that would occur under a regime of common property. Moreover, for some sets of parameters, they can also increase the utility of all agents.

Our model is a spatial generalization of the classical Levhari and Mirman (1980) example of a "fish war", although we use linear growth functions instead of strictly concave functions with a finite carrying capacity.[1] On the other hand, our model is closest to those in Herrera, Moeller and Neubert (2016) and Kaffine and Costello (2011) (also used in Costello, Quérou and Tomini, 2015, Quérou, Costello and Tomini, 2018, Costello, Nkuiya and Quérou, 2019), which use perfect Nash equilibria in the context of mobile spatially distributed stocks. There are still some relevant differences:

[1]Note that, with collapsing resource stocks there is no harm in using linear approximations of non-linear growth functions with finite steepness at zero.

(i) Unlike Herrera, Moeller and Neubert (2016), which is essentially a numerical paper, we do not focus only on steady states but we characterize equilibrium feedbacks, and we describe the whole optimal trajectory and the corresponding transition dynamics. In this way we can also analyze how welfare changes in the various specifications of the problem. Among these, we also include property rights on various parts of the sea.

(ii) Unlike the N-patch discrete time model of Kaffine and Costello (2011), the transitional dynamics of our model is richer, the behaviors of the agents depend on the whole distribution of the resource stock, and the equilibrium path does not jump to the stationary state.

Notably, our contribution provides a simple set-up where the policies we consider (TURF, reserve creation) can be directly compared against the pure common case (i.e. the absence of any regulation) so that their impact can be evaluated.

Related papers can be found in other branches of the natural resource economics literature. We mention in particular the bioeconomic model of Bhat and Huffaker (2007) on the dispersion of a small-mammal population over time and the transboundary pollution linear-quadratic differential game proposed by de Frutos and Martin Herran (2019). Besides the obvious differences with the model specifications linked to the intrinsic differences between a resource-exploitation model, wildlife control and pollution dynamics, the perfect Nash equilibria, when found, are only partially analytically characterized in these papers.

The paper proceeds as follows: in Section 2 we describe the two-player two-region model, giving in particular the definitions of Markovian Nash equilibrium. In Section 3 we study in full detail the common property scenario, first with relaxed constraints (the relaxed problem) and then with full constraints. In Section 4 we similarly analyze the scenarios in which a marine reserve (Section 4.1) and TURF (Section 4.2) are enforced, and discuss the impact of policies, comparing outcomes and overall growth rates of stock with those of the common property case (Section 4.3). In Section 5 we provide two examples of Markovian equilibria in which agents use their efforts at full capacity. In Section 6 we suggest and briefly discuss several possible extensions of the model. Section 7 contains the conclusions.

2. The model

Let us imagine that a stock of mobile resources is distributed on a given territory, partitioned into two contiguous subareas, region 1 and region 2. The stock distribution is given by a nonnegative column vector $x(t) = (x_1(t), x_2(t))'$, where $x_i(t) \geq 0$ is the biomass contained in region i ($i = 1, 2$) at time t, $t \geq 0$. Natural conditions are such that the natural resources in the two subareas have different intrinsic reproduction rates Γ_1 and Γ_2, with $\Gamma_2 \geq \Gamma_1$ (with Γ_1 not necessarily positive).

Two agents compete for the exploitation of the resource. Let $c_i^j(t)$ denote the rate of extraction of Player j in the location i at time t, with $j \in \{1, 2\}$.

Although no amount of resource can be shifted by agents from one region to the other, some living stock moves spontaneously between regions, from higher to lower biomass concentration. More precisely, we assume the diffusion process follows Fick's first law: the flow of the resource from region i to region $3 - i$ at time t is given by

$$\alpha(x_i(t) - x_{3-i}(t)), \tag{1}$$

where $\alpha > 0$ is the diffusion coefficient. The dynamics of the resource stock is then given by

$$\begin{cases} \dot{x}_1(t) = \Gamma_1 x_1(t) + \alpha(x_2(t) - x_1(t)) - c_1^1(t) - c_1^2(t), & x_1(0) = x_1^\circ \geq 0 \\ \dot{x}_2(t) = \Gamma_2 x_2(t) + \alpha(x_1(t) - x_2(t)) - c_2^1(t) - c_2^2(t), & x_2(0) = x_2^\circ \geq 0 \end{cases} \tag{2}$$

with positivity constraints on the control

$$c_i^j(t) \geq 0, \text{ for all } t \geq 0, \quad i, j \in \{1, 2\} \tag{3}$$

and on the stock

$$x_i(t) \geq 0, \text{ for all } t \geq 0, \quad i \in \{1, 2\}. \tag{4}$$

Player j chooses the strategy $c^j(t) = (c_1^j(t), c_2^j(t))'$ to maximize either the functional

$$J^j(c^j) = \int_0^{+\infty} e^{-\rho t} \frac{\left(b_1 c_1^j(t) + b_2 c_2^j(t)\right)^{1-\sigma}}{1-\sigma} dt, \tag{5}$$

where $\sigma > 0$, $\sigma \neq 1$, and $b_i \in [0, 1]$, or its logarithmic counterpart

$$J^j(c^j) = \int_0^{+\infty} e^{-\rho t} \ln\left(b_1 c_1^j(t) + b_2 c_2^j(t)\right) dt \tag{6}$$

when $\sigma = 1$. The nonnegative constants $1 - b_1$ and $1 - b_2$ represent iceberg costs, with $b_1 \leq b_2$ if region 1 is harder to reach than region 2.[2]

We also take into account the fact that extraction of a resource is more difficult where the resource is less abundant. More precisely, we consider two standard Schaefer catch functions

$$c_i^j(t) = \beta E_i^j(t) x_i(t), \tag{7}$$

where the stock $x_i(t)$ is bilinearly combined with Player j's effort $E_i^j(t)$, and β is the *catchability parameter*. Assuming Player j's total capacity for effort is finite and normalized to 1, from (7) we derive the constraints

$$\frac{c_1^j(t)}{x_1(t)} + \frac{c_2^j(t)}{x_2(t)} \leq \beta. \tag{8}$$

In the next sections we will address existence of a Markovian equilibrium for three different scenarios: (a) the case of *common property*, where each agent can decide, at any time, which regions to harvest in and by how much; (b) the case of the *reserve*, in which the regulator forbids the agents from harvesting in one of the two regions; (c) the case of *TURF*, in which each player harvests in a given region of exclusive exploitation.

2.1. **Preliminaries.** It is useful to summarize the effects of Γ_1, Γ_2 and α by introducing aggregated parameters. A natural choice is to consider the maximal eigenvalue λ and the associated eigenvector $\eta = (1, \mu)'$ of the matrix of system (2) when represented in vector form, that is

$$x'(t) = Mx(t) - C(t)e, \quad x(0) = x^\circ \equiv (x_1^\circ, x_2^\circ)'$$

where, if $\{e_1, e_2\}$ is the canonical base of column vectors for $\mathbb{R}^2$, we set

$$M := \begin{pmatrix} \Gamma_1 - \alpha & \alpha \\ \alpha & \Gamma_2 - \alpha \end{pmatrix}, \quad C(t) = \begin{pmatrix} c_1^1(t) & c_1^2(t) \\ c_2^1(t) & c_2^2(t) \end{pmatrix}, \quad e = e_1 + e_2. \tag{9}$$

The matrix M has two distinct eigenvalues

$$\lambda = \frac{1}{2}\left\{\Gamma_1 + \Gamma_2 - 2\alpha + \sqrt{4\alpha^2 + (\Gamma_2 - \Gamma_1)^2}\right\}, \quad \bar{\lambda} = \frac{1}{2}\left\{\Gamma_1 + \Gamma_2 - 2\alpha - \sqrt{4\alpha^2 + (\Gamma_2 - \Gamma_1)^2}\right\}. \tag{10}$$

[2] The quantities $(1 - b_1)$ and $(1 - b_2)$ can be also interpreted as taxes, although revenue from taxes is not part of our model.

The dominant root λ is associated to the positive eigenvector $\eta = (1, \mu)'$, and the other root $\bar{\lambda}$ to the orthogonal eigenvector $\bar{\eta} = (-\mu, 1)'$, where

$$\mu = \frac{1}{2\alpha}\left(\sqrt{4\alpha^2 + (\Gamma_2 - \Gamma_1)^2} + \Gamma_2 - \Gamma_1\right). \tag{11}$$

The dominant root λ is the von Neumann maximum rate of growth and, in this context, has a simple interpretation in terms of the technological primitives of the model, as a "weighted average" of the two intrinsic rates of growth Γ_1 and Γ_2, with the weights depending on the diffusivity coefficient α. In particular, (10) implies that $\lambda \in [\Gamma_1, \Gamma_2]$ (of course if $\Gamma_1 = \Gamma_2 \equiv \Gamma$, then $\lambda = \Gamma$). On the other hand, the component μ of the eigenvector is the relative value of the stock in region 2 in terms of the stock in region 1. Note that since $\Gamma_2 \geq \Gamma_1$ implies $\mu \geq 1$ (with $\mu > 1$ when $\Gamma_2 > \Gamma_1$), the value of the stock is higher in the most productive region. Note also that λ increases when the parameters Γ_i increase, as one might expect, and decreases when α increases. The latter may be interpreted as follows: when α is higher, fish tend to accumulate less in the more productive region, which translates into a lower overall reproduction rate λ . Similarly, μ decreases towards 1 when α increases, because a higher diffusion coefficient reduces the heterogeneity of the stocks.

Next, we need to define the set of admissible strategies for the players, taking into account that in the scenarios of a marine reserve or of TURF some of the c_i^j are muted by the social regulator. To take this into account, we define the control space $C \equiv C_1 \times C_2$ differently in the different scenarios. If (c^1, c^2), with c^j the control exerted by Player j, $c^j = (c_1^j, c_2^j) \in C_j$, then:

(a) In the common property case, $C_1 = C_2 = \mathbb{R}_+^2$.
(b) In the reserve case, when players do not harvest in region 1, we have $c_1^1 = 0 = c_1^2$, hence we set $C_1 = \{0\} \times \mathbb{R}_+$, and $C_2 = \{0\} \times \mathbb{R}_+$. When the reserve is set in region 2, C is defined symmetrically.
(c) In the TURF case, Player j harvests exclusively in region j, implying $c_2^1 = 0 = c_1^2$, so that $C_1 = \mathbb{R}_+ \times \{0\}$ and $C_2 = \{0\} \times \mathbb{R}_+$.

In order to clarify the parameter restrictions required in the ensuing analysis, in the Sections 3 and 4 we will begin by studying a relaxed problem, in which effort constraints (8) are not taken into account and positivity constraints (4) are relaxed, and only later will consider the fully constrained problem. Hence we need definitions of admissible strategy profiles and of Nash equilibria for the relaxed problem and, separately, for the fully constrained problem. In both cases we will use strategies that are stationary *Markovian*, i.e which depend only on the state $x(t)$ of the system in real time.

2.1.1. *Markovian Admissible strategy profiles for the relaxed problem.* We first define admissible Markovian strategy profiles and Markovian equilibria in the case when effort constraints (8) are not considered and (4) is relaxed to

$$x_1(t) + \mu x_2(t) \geq 0, \ \forall t \geq 0,$$

which means we assume the *value of the stock* to be non-negative, although no such requirement constrains each individual component of the state. We then define the *state space* of the relaxed problem, in all scenarios, to be the halfplane

$$S = \Big\{x = (x_1, x_2) \in \mathbb{R}^2 : \langle x, \eta \rangle = x_1 + \mu x_2 \geq 0\Big\}.$$

Note that S contains all of the positive orthant of $\mathbb{R}^2$. Note also that some constraint on the state space is needed in order to have existence of meaningful equilibria (otherwise players would choose to extract infinite amounts of resource, even from a negative stock). We denote the trajectory of (2) at time t, starting at x° and driven by the strategy profile (c^1, c^2), by $x(t; c^1, c^2, x^\circ)$. Admissible controls, at an initial state $x^\circ \in S$, are measurable functions from $[0, +\infty)$ to C that generate trajectories $x(t; c^1, c^2, x^\circ)$

which are contained in S at all times. *Markovian strategy profiles* are a subset of these admissible controls which are functions only of the current levels of stock variables. The formal definition follows.

Definition 2.1 (Markovian Admissible strategy profiles for the relaxed problem) *Consider a given initial state $x^\circ = (x_1^\circ, x_2^\circ) \in S$. We say that a pair of continuous functions $\psi := (\psi^1, \psi^2) = ((\psi_1^1, \psi_2^1), (\psi_1^2, \psi_2^2)) \colon S \to \mathbb{R}_+^2 \times \mathbb{R}_+^2$ is an* admissible (stationary) Markovian strategy profile for the relaxed problem at x° *if:*

(i) *$\psi(x) \in C$, for all $x \in S$, and $\psi_i^j(x) = 0$ when $\langle x, \eta\rangle = 0$, for all $i, j = 1, 2$.*
(ii) *the equation (2) with $c_i^j(t)$ replaced by $\psi_i^j(x(t))$, i.e.*

$$\begin{cases} \dot{x}_1(t) = \Gamma_1 x_1(t) + \alpha(x_2(t) - x_1(t)) - \psi_1^1(x(t)) - \psi_1^2(x(t)), & x_1(0) = x_1^\circ \\ \dot{x}_2(t) = \Gamma_2 x_2(t) + \alpha(x_1(t) - x_2(t)) - \psi_2^1(x(t)) - \psi_2^2(x(t)), & x_2(0) = x_2^\circ \end{cases} \tag{12}$$

has a unique solution $x^\psi(\cdot)$;

We denote by $\mathcal{M}^{rel}(x^\circ)$ the set of all admissible Markov strategy profiles for the relaxed problem at x°.

In this definition we have required the feedback strategy to be null at the boundary of S rather than directly requiring that $x(t) \in S$ at all times. In fact this restriction is enough to keep trajectories in S at all times, as implied by the following lemma.

Lemma 2.2 *Let $x^\circ \in S$ be a given initial state. Suppose that $\psi_i^j \colon S \to \mathbb{R}_+$ for $j, i = 1, 2$ are continuous functions such that the system* (13) *has a unique solution $x^\psi(\cdot)$. Assume also that $\psi_i^j(x) = 0$ for all i, j, for $x \in S$ satisfying $\langle x, \eta\rangle = 0$. Then $x^\psi(t) \in S$, for all $t \geq 0$.*

Definition 2.3 (Markovian Nash equilibrium for the relaxed problem) *Let $x^\circ := (x_1^\circ, x_2^\circ) \in S$ be a given initial state, and let $\psi \in \mathcal{M}^{rel}(x^\circ)$. We say that $\psi = (\psi^1, \psi^2)$ is a* (stationary) Markovian Nash equilibrium for the relaxed problem at x° *if:*

(i) *The control $\psi^1(x)$ is optimal for the problem of Player 1 given by: the state equation* (2) *where Player 2 chooses $\psi^2(x)$; the constraints* (3)*; the functional $J^1(c^1)$ given by* (5) *(or, in alternative,* (6)*), to be maximized over the set of admissible controls*

$$\mathcal{C}^{\psi^2}(x^\circ) = \left\{c^1 \colon [0, +\infty) \to C_1 \ : \ x(t; c^1, \psi^2(x); x^\circ) \in S, \forall t \geq 0\right\}.$$

In formulas

$$J^1(\psi^1(x)) \geq J^1(c^1), \quad \forall c^1 \in \mathcal{C}^{\psi^2}(x^\circ).$$

(ii) *The symmetric statement holds when the roles of Player 1 and 2 are exchanged.*

Remark 2.4 In both these definitions, the feedback strategy profile ψ takes values in C, so that some of its components have to be considered identically null, consistent with the cases (a)–(c) in the description of the strategy profile set C. □

2.1.2. *Markovian Admissible strategy profiles for the constrained problem.* We now consider the original fully constrained problem, where players maximize the functionals given by (5) (or by (6)) under the state equation (2), with all constraints (3), (4) and (8). We define admissible Markovian strategy profiles and Markovian equilibria accordingly.

Definition 2.5 (Markovian Admissible strategy profiles) *Let $x^\circ := (x_1^\circ, x_2^\circ) \in \mathbb{R}_+^2$ be a given initial state. We say that a pair of continuous functions $\psi := (\psi^1, \psi^2) = ((\psi_1^1, \psi_2^1), (\psi_1^2, \psi_2^2)) \colon \mathbb{R}_+^2 \to \mathbb{R}_+^2 \times \mathbb{R}_+^2$ is an* admissible (stationary) Markovian strategy profile at x° *if:*

(i) *$\psi(x) \in C$ and $\frac{\psi_1^j(x)}{\beta x_1} + \frac{\psi_2^j(x)}{\beta x_2} \leq 1$, for $j = 1, 2$, for all $x \in \mathbb{R}_+^2$;*

(ii) *the equation (2) with* $c_i^j(t)$ *replaced by* $\psi_i^j(x(t))$, *i.e.*

$$\left\{ \begin{array}{ll} \dot{x}_1(t) = \Gamma_1 x_1(t) + \alpha(x_2(t) - x_1(t)) - \psi_1^1(x(t)) - \psi_1^2(x(t)), & x_1(0) = x_1^\circ \\ \dot{x}_2(t) = \Gamma_2 x_2(t) + \alpha(x_1(t) - x_2(t)) - \psi_2^1(x(t)) - \psi_2^2(x(t)), & x_2(0) = x_2^\circ \end{array} \right. \tag{13}$$

has a unique solution $x^\psi(\cdot)$;

We denote by $\mathcal{M}(x^\circ)$ *the set of all admissible Markov strategy profiles for the problem at* x°.

In $\mathcal{M}(x^\circ)$ we have not explicitly required that $x^\psi(\cdot)$ satisfy positivity constraints (4). Nonetheless, as we prove in the next lemma, the effort constraints (8) imply the positivity of the trajectory when the system starts from a positive initial state.

Lemma 2.6 *Let* $x^\circ \in \mathbb{R}^2_+$ *be a given initial state. Suppose that* $\psi_i^j \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ *for* $j, i = 1, 2$ *are continuous functions such that* $\frac{\psi_1^j(x)}{\beta x_1} + \frac{\psi_2^j(x)}{\beta x_2} \leq 1$ *for* $j = 1, 2$ *and for all* $x \in \mathbb{R}^2_+$, *and that the system* (13) *has a unique solution* $x^\psi(\cdot)$. *Then, for all* $t \geq 0$, $x_i^\psi(t) \geq 0$, *for* $i = 1, 2$.

Definition 2.7 **(Markovian Nash equilibrium)** *Let* $x^\circ := (x_1^\circ, x_2^\circ) \in \mathbb{R}^2_+$ *be a given initial state, and let* $\psi \in \mathcal{M}(x^\circ)$. *We say that* $\psi = (\psi^1, \psi^2)$ *is a* (stationary) Markovian Nash equilibrium *at* x° *for the constrained problem if the following two conditions hold:*

(*i*) *Consider the control problem for Player 1 given by: the state equation* (2) *where Player 2 chooses* $\psi^2(x)$; *the functional* $J^1(c^1)$ *described by* (5) *(or* (6)*). Then* $\psi^1(x)$ *maximizes* $J^1(c^1)$ *over the set of admissible controls*

$$\mathcal{C}^{\psi^2}(x^\circ) = \left\{ c^1 \colon [0, +\infty) \to C_1 \ : \ x(t) \in \mathbb{R}^2_+, \ \textit{and} \ \frac{c_1^1(t)}{\beta x_1(t)} + \frac{c_2^1(t)}{\beta x_2(t)} \leq 1, \forall t \geq 0 \right\},$$

where $x(t) \equiv x(t; c^1, \psi^2(x); x^\circ)$. *In formulas*

$$J^1(\psi^1(x)) \geq J^1(c^1), \quad \forall c^1 \in \mathcal{C}^{\psi^2}(x^\circ).$$

(*ii*) *The symmetric statement holds true when roles of Player 1 and Player 2 are exchanged.*

Remark 2.8 The next sections will describe Markovian Nash equilibria only for certain choices of initial states x°, namely those in a cone contained in the first orthant (for example, cone $K_1(z)$ defined in (24), for the case of the common property), which depends on the choice of the parameters. That does not mean that equilibria described by Definition 2.7 are local, for instance in the sense described by Dockner and Wagener (2014). In fact we do not impose any restriction (in terms of admissible states and controls, or any additional requirements) to the orginal economic problem with its effort and positivity constraints. In particular, when maximizing their welfare, players consider viable all strategies in $\mathcal{M}(x_0)$, even those that would drive the system out of the cone (as clarified along the proof of Theorem 3.8 in the appendix), but finally choose a strategy that keeps the state in the cone at all times. □

3. Common Property, Low-Intensity Harvesting

The scenario considered here is when players are free to fish in any region. We recall that, in such a case, the space of strategy profiles is $C = \mathbb{R}^2_+ \times \mathbb{R}^2_+$. We initially consider the relaxed problem, described by (2) and (3) where each player is maximizing (5), and prove the existence of a Markov equilibrium. In two separate subsections, we discuss how constraints (4) and (8) modify the result.

Theorem 3.1 (Markovian Nash equilibrium for the common, relaxed problem) *Consider the relaxed problem described by* (2) *and* (3) *where each player is maximizing* (5). *Consider* $x^\circ = (x_1^\circ, x_2^\circ) \in S$, *set*

$z = \frac{\rho-(1-\sigma)\lambda}{2\sigma-1}$, *and assume* $z > 0$. *When* $\mu > \frac{b_2}{b_1}$, *the strategy profile* $\psi \in \mathcal{M}^{rel}(x^\circ)$ *defined by*

$$\begin{aligned} \psi_1^1(x) &= z\left(x_1 + \mu x_2\right), \qquad \psi_2^1(x) \equiv 0 \\ \psi_1^2(x) &= z\left(x_1 + \mu x_2\right), \qquad \psi_2^2(x) \equiv 0 \end{aligned} \tag{14}$$

for all $x \equiv (x_1, x_2) \in S$ *is a Markovian Nash equilibrium at* x° *in the sense of Definition 2.3. Moreover, the utilities of the players at equilibrium are, respectively,* $v^1(x_1^\circ, x_2^\circ), v^2(x_1^\circ, x_2^\circ)$, *where*

$$v^2(x_1, x_2) = v^1(x_1, x_2) = b_1^{1-\sigma}\ \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)}. \tag{15}$$

Similarly, if $\mu < \frac{b_2}{b_1}$, *the equilibrium is given by*

$$\begin{aligned} \psi_1^1(x) &\equiv 0, \qquad \psi_2^1(x) = \tfrac{z}{\mu}\left(x_1 + \mu x_2\right) \\ \psi_1^2(x) &\equiv 0, \qquad \psi_2^2(x) = \tfrac{z}{\mu}\left(x_1 + \mu x_2\right) \end{aligned}$$

and

$$v^2(x_1, x_2) = v^1(x_1, x_2) = \left(\frac{b_2}{\mu}\right)^{1-\sigma} \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\rho)} .$$

In both cases, the trajectory of the system at the equilibrium satisfies

$$x_1(t) + \mu x_2(t) = \langle x(t), \eta\rangle = (x_1^\circ + \mu x_2^\circ)e^{gt} \tag{16}$$

with $g = \lambda - 2z = \frac{\lambda - 2\rho}{2\sigma - 1}$.

Remark 3.2 Equation (16) implies that the equilibrium strategies c_i^j and utilities v^j depend merely on the projection of $x(t)$ along the direction of the eigenvector η. To explain the shape of the equilibrium extraction policies and of the coefficient z, it is useful to compare the equilibrium in Theorem 3.1 with the analogous linear Markov equilibrium arising in the simpler homogeneous stock AK model, where the state equation is given by

$$\dot{x} = \Gamma x - c_1 - c_2. \tag{17}$$

It is well known for this model that if Player 2 uses the linear strategy $c_2 = z_2 x$, where z_2 is a positive constant, then Player 1 perceives the resource stock growth at rate $\Gamma - z_2$ and reacts by choosing

$$z_1 = \frac{c_1}{x} = \frac{\rho - (1-\sigma)(\Gamma - z_2)}{\sigma}. \tag{18}$$

The reaction function of Player 2 to a linear strategy $c_1 = z_1 x_1$ of Player 1 is symmetric. Hence if the elasticity of intertemporal substitution $\frac{1}{\sigma} \neq 2$, conditions for existence of the symmetric equilibrium are

$$z \equiv z_1 = z_2 = \frac{\rho - (1-\sigma)\Gamma}{2\sigma - 1}, \text{ with } z > 0. \tag{19}$$

If instead $\frac{1}{\sigma} = 2$, then the slope of the reaction function (18) is 1 and a symmetric equilibrium (actually a continuum of symmetric equilibria) exists only in the singular case $\Gamma = 2\rho$.

Note that, by (18), $z > 0$ is equivalent to $\rho - (1-\sigma)(\Gamma - z) > 0$, which is a necessary and sufficient condition for the existence of a solution for the control problem of any player (this is the standard condition for the existence of a solution of an infinite horizon control problem with an isoelastic utility function and a stock growing linearly at the rate $\Gamma - z$). When there is only one agent (or, equivalently, a social planner), this existence condition reduces to $\rho - (1-\sigma)\Gamma > 0$, implying that the set of parameters where a linear Markov equilibrium exists changes with the number of players. In particular, both the social planner solution and the Markov equilibrium for the two-player game exist if $\rho - (1-\sigma)\Gamma > 0$ and $\frac{1}{\sigma} < 2$, the social planner solution exists, while the two-player Markov equilibrium does not exist if $\rho - (1-\sigma)\Gamma > 0$ and $\frac{1}{\sigma} > 2$, and, finally, the two-player Markov equilibrium exists, but the optimal solution for the social planner does not, if $\rho - (1-\sigma)\Gamma < 0$ and $\frac{1}{\sigma} > 2$. With two players, the last two inequalities characterize the interior equilibrium in Proposition 2 on pages 30-31 of the slightly different model in Lane and Tornell (1999) (see also Dockner, Jorgensen, Van Long and Sorger, 2000, section 12.1 for the case $\Gamma = 0$). As in Lane and Tornell (1999), a "voracity effect" arises when these two conditions are satisfied.

Theorem 3.1 shows that, with two appropriate qualifications, the interior linear Markov equilibrium of the scalar model can be extended to the two-region model where the stock is spatially distributed. First, since the resource is non-homogeneous across space, a price is needed to aggregate the stock. The component μ of the dominant eigenvector is the price of the stock in region 2 in terms of the stock in region 1 that allows the aggregation, so the policy function turns out to be linear in the value of the stock $x_1(t) + \mu x_2(t)$. Second, in the two-location setting the one-sector "productivity of the stock" Γ is substituted by the von Neumann maximum rate of growth, which in our single production framework coincides with the dominant eigenvalue λ of the production matrix M (see e.g., Freni, Gozzi and Salvadori, 2006). While Theorem (3.1) does not cover the case $\frac{1}{\sigma} = 2$ and $\lambda = 2\rho$ (where z is indeterminate), the existence of a continuum of symmetric equilibria can be established in this case via a straightforward extension of the argument used above for the scalar model. □

Remark 3.3 The fact that both players, depending on the ratio $\frac{b_2}{b_1}$, fish in just one region (the same for both), follows from the assumption that the catches of the two regions are perfect substitutes. So when $\mu = \frac{b_2}{b_1}$, Player j is indifferent among all the strategies (c_1^j, c_2^j) such that

$$\psi_1^j(x) + \mu\psi_2^j(x) = z\,(x_1 + \mu x_2)\,.$$

A linear symmetric Markov equilibrium is then

$$\psi_1^j(x) = \theta z\,(x_1 + \mu x_2)$$

$$\psi_2^j(x) = (1-\theta)\frac{z}{\mu}\,(x_1 + \mu x_2)$$

for any $\theta \in [0,1]$ (possibly depending on time). The same argument applies for the logarithmic utility discussed in Remark 3.4. We also note that in the case $b_1 = b_2$ the theorem establishes that the equilibrium policy is such that both players are fishing only in the least productive region 1 where fish reproduces at rate Γ_1 while, at the same time, fish reproduces undisturbed in region 2 at the higher rate $\Gamma_2 > \Gamma_1$. □

Remark 3.4 For the logarithmic utility (6) the results of Theorem 3.1 hold true with due changes. For instance, when $\mu > \frac{b_2}{b_1}$, the equilibrium is given by

$$\begin{array}{ll} \psi_1^1(x) = \rho\,(x_1 + \mu x_2)\,, & \psi_2^1(x) \equiv 0 \\ \psi_1^2(x) = \rho\,(x_1 + \mu x_2)\,, & \psi_2^2(x) \equiv 0, \end{array} \tag{20}$$

and the utility is $v^2(x_1,x_2) = v^1(x_1,x_2) = \frac{1}{\rho}\left(\frac{\lambda-2\rho}{\rho} + \ln(b_1\rho) + \ln(x_1+\mu x_2)\right)$. □

3.1. **Trajectories and constraints.** We now discuss the behaviour at equilibrium of the resource stock $x(t)$ of the relaxed problem. Equation (2) becomes $x(t) = Bx(t)$, $x(0) = x^\circ$, with $B = B_1$ if $\mu > \frac{b_2}{b_1}$, and $B = B_2$ if $\mu < \frac{b_2}{b_1}$, and

$$B_1 = \begin{pmatrix} \Gamma_1 - \alpha - 2z & \alpha - 2z\mu \\ \alpha & \Gamma_2 - \alpha \end{pmatrix}, \quad B_2 = \begin{pmatrix} \Gamma_1 - \alpha & \alpha \\ \alpha - \frac{2z}{\mu} & \Gamma_2 - \alpha - 2z \end{pmatrix}. \tag{21}$$

By direct computation, one may show that both B_1 and B_2 have the same eigenvalues: $\bar\lambda$, associated to the eigenvalue $\bar\eta$ (defined below equation (10)), and $g = \lambda - 2z$, associated to an eigenvector $(u_1, 1)'$, a perturbation of η through extraction, as detailed below.

Lemma 3.5 *Under the assumptions of Theorem 3.1, consider the equilibrium trajectory described there, starting at $x^\circ \in S$. Then*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = k_1 \begin{pmatrix} -\mu \\ 1 \end{pmatrix} e^{\bar{\lambda} t} + k_2 \begin{pmatrix} u_1 \\ 1 \end{pmatrix} e^{gt}, \tag{22}$$

where k_1, k_2 are constants determined by the initial condition $x(0) = x^\circ$, while $u_1 = \frac{1}{\mu} - \frac{2z}{\alpha}$ when $\mu > \frac{b_2}{b_1}$, and $u_1 = \left(\mu - \frac{2z}{\alpha}\right)^{-1}$ when $\mu < \frac{b_2}{b_1}$.

3.1.1. *Positivity constraints on the state.* Note that $x_1(t) + \mu x_2(t) \geq 0$ at all times as a consequence of (16), although nothing has been said, so far, on the positivity of each single components $x_i(t)$. The results are summarized in the proposition below.

Proposition 3.6 *Under the assumptions of Theorem 3.1, consider the equilibrium trajectory described there, starting at $x^\circ \in \mathbb{R}^2_+$. Then $x(t)$ converges in time towards the direction of the eigenvector $(u_1, 1)$, where u_1 is defined in Lemma 3.5. Moreover it satisfies* (4) *if and only if $u_1 \geq 0$.*

The proof of this proposition is given in the appendix, and shows that $\frac{x_1(\cdot)}{x_2(\cdot)}$ converges monotonically to the value u_1. Note that the proof of positivity of $x(t)$ can be also deduced by the fact that B_1 (or B_2) is a Metzler matrix (i.e. has all nonnegative elements except those on the main diagonal) if and only if $u_1 \geq 0$, and that the property characterizes positive systems, i.e. systems with nonnegative trajectories for all choices of a nonnegative initial datum x° (see for instance Farina and Rinaldi, 2000, Chapter 2). Note also that $u_1 \geq 0$ is equivalent to requiring that $(u_1, 1)$ is in the positive orthant. The behaviour of trajectories starting from a x° in the first orthant is depicted in Figure 1.

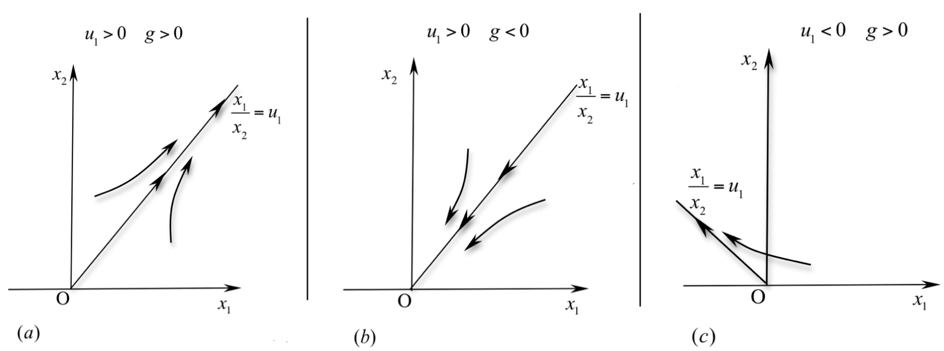

FIGURE 1. In (a) and (b) positivity constraints are satisfied, as $\alpha - 2\mu z > 0$, but while in (a) g is positive, so there is growth (to infinity) of both $x_1(t)$ and $x_2(t)$, in (b) we have $g < 0$ implying that the stocks in each region are asymptotically decreasing to 0 in time. In (c), where $\alpha - 2\mu z < 0$ after a certain time, the trajectory leaves the positive orthant.

3.1.2. *Effort constraints on the state.* We further analyze the role of effort constraints (8) on the existence of a Markovian equilibrium. For simplicity we study extensively only the case when $\mu > \frac{b_2}{b_1}$, and $(x_1^\circ, x_2^\circ) \in \mathbb{R}^2_+$. We start by noting that the equilibrium trajectory $(x_1(t), x_2(t))$ described by Theorem 3.1 satisfies (8) if and only if it satisfies

$$(\beta - z)x_1(t) \geq \mu\, z\, x_2(t), \ \forall t \geq 0, \tag{23}$$

meaning that the entire trajectory needs to be contained in the cone

$$K_1(z) = \left\{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ (\beta - z)x_1 \geq \mu z x_2\right\}. \tag{24}$$

A necessary and sufficient condition for that is

$$(u_1, 1) \in K_1(z) \quad \text{and} \quad x^\circ \in K_1(z), \tag{25}$$

as we specify next. Note that the first inequality can be satisfied only when $u_1 \geq 0$, consistent with Lemma 2.6 and Proposition 3.6. Moreover the requirements in (25), interpreted in the phase plane (x_1, x_2) along time, say that a necessary and sufficient condition for the trajectory to be entirely contained in $K_1(z)$ is that both the eigenvector $(u_1, 1)'$ and the initial datum (x_1°, x_2°) be contained in the cone $K_1(z)$ (see Figure 2).

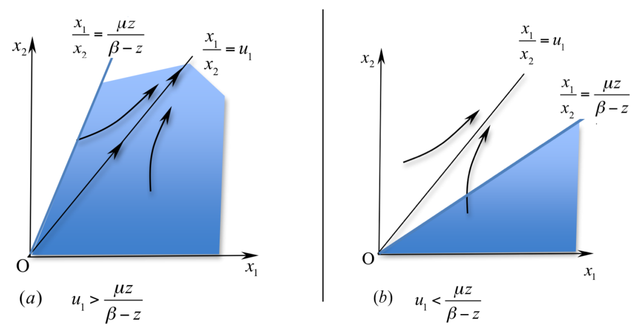

FIGURE 2. Behaviour of trajectories for a given z in the case $\mu > b_2/b_1, g > 0$. The cone $K_1(z)$ is shaded in blue. In (a) $(u_1, 1) \in K_1$, and the trajectory is entirely contained in $K_1(z)$, while in (b) $(u_1, 1) \notin K_1$ and the trajectory leaves $K_1(z)$ after a certain time, violating the effort constraints. The trajectory in both cases converges along the direction of $(u_1, 1)$

Since u_1 is itself a function of z, one may want to establish which values of z (25) is satisfied for. It turns out that the property holds true in an interval $(0, z_1^*]$ as specified in Proposition 3.7 below. The case $\mu < \frac{b_2}{b_1}$ can be similarly discussed, with initial condition taken in the cone

$$K_2(z) = \left\{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ (\beta - z)x_2 \geq \mu^{-1} z \, x_1\right\}, \tag{26}$$

and $z \in (0, z_2^*]$, where z_2^* is also defined in Proposition 3.7.

Proposition 3.7 *Let* $z = \frac{\rho - (1-\sigma)\lambda}{2\sigma - 1}$ *and set*

$$\begin{aligned} z_1^* &:= \tfrac{1}{4}\left[2\beta + \sqrt{(\Gamma_2 - \Gamma_1)^2 + 4\alpha^2} - \sqrt{(\Gamma_2 - \Gamma_1 + 2\beta)^2 + 4\alpha^2}\right] \\ z_2^* &:= \tfrac{1}{4}\left[2\beta + \sqrt{(\Gamma_2 - \Gamma_1)^2 + 4\alpha^2} - \sqrt{(\Gamma_2 - \Gamma_1 - 2\beta)^2 + 4\alpha^2}\right]. \end{aligned} \tag{27}$$

Then $0 \leq z_1^* \leq \alpha/(2\mu)$ *and* $0 \leq z_2^* \leq (\mu\alpha)/2$. *Moreover, when* $\mu > \frac{b_2}{b_1}$ *(respectively,* $\mu < \frac{b_2}{b_1}$*), the trajectory at equilibrium described in Theorem 3.1 satisfies constraints* (4) *and* (8) *if and only if* $z \in (0, z_1^*]$ *(respectively,* $z \in (0, z_2^*]$*) and* $x^\circ \in K_1(z)$ *(respectively,* $x^\circ \in K_2(z)$*)* .

3.2. **Markovian Nash Equilibria, in Low-intensity Harvesting.** We have highlighted through Proposition 3.7 how, for some values of the parameters $\Gamma_1, \Gamma_2, \alpha, \sigma, \rho$, and β, the strategies ψ described in Theorem 3.1 and the associated trajectories satisfy both positivity and effort constraints. We then refer to these sets of parameters as the case of *low-intensity harvesting*, meaning that players choose to extract below the maximum effort described by (8), though free to choose otherwise.

To better describe the parameters identifying a low intensity of harvest, Figure 3 represents, for the case $\mu > \frac{b_2}{b_1}$, the pairs $(\rho, 1-\sigma)$ for which, in the respective highlighted areas:

(a) $z = \frac{\rho - (1-\sigma)\lambda}{2\sigma - 1} > 0$;

(b) $0 < z \le \alpha/(2\mu)$, with positivity constraints on the state enabled;
(c) $0 < z \le z_1^*$, with both state and effort constraints enabled.

Note that all equations of type $z = c$, for a given $c \in \mathbb{R}$, can be written in terms of σ and ρ as $\rho - (1-\sigma)\lambda = (2\sigma - 1)c$, so the lines $z = 0$, $z = \alpha/(2\mu)$ and $z = z_1^*$ all cross at $(\lambda/2, 1/2)$ for all β. As β is a bound on the extraction effort, a large β reduces the impact of the effort constraints. Indeed, when β decreases, then the line $z = z_1^*$ rotates clockwise, coinciding with $z = 0$ at limits, as β tends to 0^+. On the other hand, when β increases, then $z = z_1^*$ rotates counterclockwise, coinciding with $z = \alpha/(2\mu)$ at limits, as β tends to $+\infty$.

In Figure 3(c), the upper shaded triangle is the set of parameters inducing a *voracity effect*, described in detail in Section 4.4.

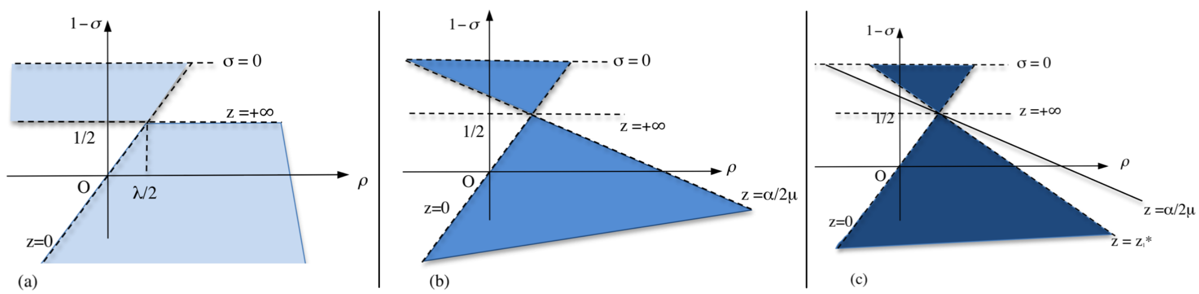

FIGURE 3

We can now state existence of a Markovian Nash equilibrium in the problem of Section 2.

Theorem 3.8 (Markovian Nash equilibrium for the common, low-intensity harvesting) *Consider the problem described by* (2) *and* (3)*, with constraints* (4) *and* (8)*, and where each player is maximizing* (5)*. Consider an initial state* $x^\circ \ge 0$*, set* $z = \frac{\rho-(1-\sigma)\lambda}{2\sigma-1}$*. Consider the following strategy profiles* $\psi \in \mathcal{M}(x^\circ)$*:*

(*i*) *When* $\mu > b_1/b_2$*,* $z \in (0, z_1^*]$ *and* $x^\circ \in K_1(z)$*, with* z_1^* *given by* (27) *and* $K_1(z)$ *by* (24)*,*

$$\begin{aligned} \psi_1^1(x) &= z\,(x_1 + \mu x_2) \wedge \beta x_1, \qquad & \psi_2^1(x) &\equiv 0 \\ \psi_1^2(x) &= z\,(x_1 + \mu x_2) \wedge \beta x_1, \qquad & \psi_2^2(x) &\equiv 0. \end{aligned}$$

(*ii*) *When* $\mu < b_1/b_2$*,* $z \in (0, z_2^*]$ *and* $x^\circ \in K_2(z)$*, with* z_2^* *given by* (27) *and* $K_2(z)$ *by* (26)*,*

$$\begin{aligned} \psi_1^1(x) &= 0, \qquad & \psi_2^1(x) &= \tfrac{z}{\mu}\,(x_1 + \mu x_2) \wedge \beta x_2 \\ \psi_1^2(x) &= 0, \qquad & \psi_2^2(x) &= \tfrac{z}{\mu}\,(x_1 + \mu x_2) \wedge \beta x_2. \end{aligned} \tag{28}$$

Then ψ *is a Markovian Nash equilibrium at* x°*, in the sense of Definition 2.7. The utilities of players at equilibrium are again those described in Theorem 3.1.*

Remark 3.9 When $\mu = \frac{b_2}{b_1}$ (see also Remark 3.3), since several indifferent strategies are possible, the constraint on the initial datum is less stringent than in Theorem 3.1 and in particular a feasible equilibrium in the described set exists as long as either $0 < z \le z_1^*$ or $0 < z \le z_2^*$. A similar argument applies in the case of logarithmic utility discussed in Remark 3.4. □

4. Policy Enhancement, Low-Intensity Harvesting

We now analyze the scenarios where a marine reserve or TURF are enforced, and derive an explicit formula for the equilibria. We then discuss the impact of these policies, comparing outcomes and overall growth rates of stock with those in the common property scenario. We also discuss how a technology shock or reduced mobility of the resource may impact the overall growth rate of the resource, particularly for sets of parameters that incentivate players to become voracious.

4.1. **The Reserve Case.** In this scenario, one region is kept as a reserve where fishing is forbidden, and the control space described in Section 2 has null components corresponding to that region. We established in Section 3 that, when free to fish anywhere, players would choose only one region, the same for both: region 1 if $\mu > b_1/b_2$ and region 2 if $\mu < b_1/b_2$. In the theorem below we analyze the case in which the reserve is set in the region where players would prefer to fish when free to choose. Indeed, if the reserve were set in the other region, the equilibrium would coincide with that already described in Theorem 3.1 (for the relaxed problem) and Theorem 3.8 (for the fully constrained problem). In particular, in the following theorem, we assume that $\mu > b_1/b_2$, the reserve is set in region 1, and the control space is $C = C_1 \times C_2$ where $C_1 = C_2 = \{0\} \times \mathbb{R}_+$. The case of $\mu < b_1/b_2$ and the reserve set in region 2 yields similar results. We first treat the case in which initial data and trajectories are entirely contained in S, and then extend the results to the fully constrained problem.

Theorem 4.1 (Markovian Nash equilibrium for the reserve, relaxed problem) *Consider the relaxed problem described by* (2), (3) *where each player is maximizing* (5). *Assume that harvesting is forbidden in region 1, namely $C = C_1 \times C_2$ where $C_1 = C_2 = \{0\} \times \mathbb{R}_+$, and that $\mu > \frac{b_2}{b_1}$, $z = \frac{\rho-(1-\sigma)\lambda}{2\sigma-1}$ with $z > 0$, and $x^\circ \in S$. Then the strategy profile $\psi \in \mathcal{M}^{rel}(x^\circ)$ defined as*

$$\begin{array}{ll} \psi_1^1(x) = 0, & \psi_2^1(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \\ \psi_1^2(x) = 0, & \psi_2^2(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \end{array} \tag{29}$$

is a (symmetric) Markovian Nash equilibrium at x°, in the sense of Definition 2.3. At equilibrium, the utilities of the respective players are $v^1(x_1^\circ, x_2^\circ)$ and $v^2(x_1^\circ, x_2^\circ)$, where

$$v^2(x_1, x_2) = v^1(x_1, x_2) = \left(\frac{b_2}{\mu}\right)^{1-\sigma} \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma (1-\rho)} .$$

The proof, similar to that of Theorem 3.1, is omitted. Extensions to the case of logarithmic utility and the case $\mu = \frac{b_2}{b_1}$ are possible, yielding results similar to those mentioned in Remarks 3.4 and 3.3, respectively.

Remark 4.2 The trajectory at equilibrium described in Theorem 4.1 satisfies the equation $x(t) = B_2 x(t)$, where B_2 is defined in (21). Hence the trajectory satisfies (4) if and only if $\mu\alpha - 2z \geq 0$, as discussed in Lemma 3.6. In addition, the trajectory satisfies (8) (and consequently (4)) for those z such that $x(t)$ is entirely contained in $K_2(z)$. The analysis of the fully constrained problem is reported in Theorem 4.3 below.

Theorem 4.3 (Markovian Nash equilibrium for the reserve, low-intensity harvesting) *Consider the problem described in Theorem 4.1 with the addition of the constraints* (4) *and* (8). *Assume also $z \in (0, z_2^*]$ and $x^\circ \in K_2(z)$, with z_2^* given by* (27) *and $K_2(z)$ by* (26). *Then the strategy profile $\psi \in \mathcal{M}(x^\circ)$ defined by*

$$\begin{array}{ll} \psi_1^1(x) = 0, & \psi_2^1(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \wedge \beta x_2 \\ \psi_1^2(x) = 0, & \psi_2^2(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \wedge \beta x_2, \end{array} \tag{30}$$

is a (symmetric) Markovian Nash equilibrium at x°, in the sense of Definition 2.7. The utilities of the players at equilibrium are those described in Theorem 4.1.

4.2. **The TURF Case.** In the third scenario we consider, each player has the right of exclusive exploitation of one of the two regions. We assume that Player j harvests exclusively in region j, so $c_2^1 = 0 = c_1^2$, and that the control space is $C = C_1 \times C_2$, with $C_1 = \mathbb{R}_+ \times \{0\}$ and $C_2 = \{0\} \times \mathbb{R}_+$. As in the previous cases, we treat the relaxed problem before the fully constrained one.

Theorem 4.4 *Consider the relaxed problem described by* (2) *and* (3) *with each player maximizing* (5). *Assume* $C = C_1 \times C_2$ *where* $C_1 = \mathbb{R}_+ \times \{0\}$, $C_2 = \{0\} \times \mathbb{R}_+$, $z = \frac{\rho-(1-\sigma)\lambda}{2\sigma-1}$, *with* $z > 0$, *and* $x^\circ \in S$. *Then the strategy profile* $\psi \in \mathcal{M}^{rel}(x^\circ)$ *defined as*

$$\begin{array}{ll} \psi_1^1(x) = z\,(x_1 + \mu x_2)\,, & \psi_2^1(x) = 0 \\ \psi_1^2(x) = 0, & \psi_2^2(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \end{array} \tag{31}$$

for all $x = (x_1, x_2) \in S$, *is a Markovian Nash equilibrium at* x°, *in the sense of Definition 2.3. At equilibrium, the utilities of the respective players are* $v^1(x_1^\circ, x_2^\circ)$ *and* $v^2(x_1^\circ, x_2^\circ)$ *where*

$$v^1(x_1, x_2) = b_1^{1-\sigma}\,\frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)}, \qquad v^2(x_1, x_2) = \left(\frac{b_2}{\mu}\right)^{1-\sigma}\frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)} \tag{32}$$

Moreover, the value of the stock is given by

$$\langle x(t), \eta\rangle = x_1(t) + \mu x_2(t) = (x_1^\circ + \mu x_2^\circ)e^{gt}, \quad \forall t \geq 0, \tag{33}$$

where $g = \lambda - 2z$.

We now consider the trajectory $x(t)$ at the equilibrium described in Theorem 4.4 and discuss for which values of the parameters constraints (4) and (8) are satisfied. Such a trajectory satisfies $x(t) = B_3 x(t)$ and $x(0) = x^\circ$, with

$$B_3 = \begin{pmatrix} \Gamma_1 - \alpha - z & \alpha - \mu z \\ \alpha - \frac{z}{\mu} & \Gamma_2 - \alpha \end{pmatrix}. \tag{34}$$

Then, arguing as in the case of common property, we derive

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = k_1 \begin{pmatrix} -\mu \\ 1 \end{pmatrix} e^{\bar{\lambda} t} + k_2 \begin{pmatrix} u_1 \\ 1 \end{pmatrix} e^{gt}, \tag{35}$$

where k_1, k_2 are computed by means of the initial condition $x(0) = x^\circ$, and

$$u_1 \equiv u_1(z) = \frac{\alpha - z\mu}{\alpha\mu - z}.$$

Note that $x(t) = B_3 x(t)$ is a positive system if and only if $\alpha - \mu z \geq 0$ and $\alpha - z/\mu \geq 0$, that is, if and only if $z \leq \alpha/\mu$. This leads to the following proposition, whose proof is similar to that of Proposition 3.6, with similar implications, and therefore omitted.

Proposition 4.5 *Under the assumptions of Theorem 4.4 , consider the equilibrium trajectory described there, which starts at* $x^\circ \in \mathbb{R}_+^2$. *Then* $x(t)$ *converges in time towards the direction of the eigenvector* $(u_1, 1)'$, *and satisfies* (4) *if and only if* $z \leq \frac{\alpha}{\mu}$.

Now note that the effort constraints (8) can be expressed as

$$\frac{\mu z}{\beta - z}\,x_2(t) \leq x_1(t) \leq \frac{\mu(\beta - z)}{z}\,x_2(t), \quad \forall t \geq 0 \tag{36}$$

meaning that the trajectory $x(t)$ must be contained at all times in the cone

$$K_3(z) = \left\{(x_1, x_2) \in \mathbb{R}_+^2 : \frac{\mu z}{\beta - z}x_2 \leq x_1 \leq \frac{\mu(\beta - z)}{z}x_2\right\}. \tag{37}$$

Note that $K_3(z) = K_1(z) \cap K_2(z) \neq \emptyset$ if and only if $z \leq \beta/2$, and that $K_3(z)$ coincides with the positive semiaxis x_1 as $z \to 0^+$. Also, since trajectories $x(t)$ starting at $x^\circ \in \mathbb{R}_+^2$ converge along the direction of $(u_1(z), 1)$, (36) is equivalent to choosing z so that both $(u_1(z), 1)$ and x° belong to $K_3(z)$, as specified in the next lemma.

Proposition 4.6 *Define*

$$z_3^* := \frac{1}{4}\left[\beta + \sqrt{(\Gamma_2 - \Gamma_1)^2 + 4\alpha^2} - \sqrt{(\Gamma_2 - \Gamma_1 + \beta)^2 + 4\alpha^2 - 4\alpha\beta/\mu}\right]. \tag{38}$$

Then $z_3^* \in (0, \frac{\alpha}{\mu} \wedge \frac{\beta}{2}]$. Moreover, the trajectory at equilibrium described in Theorem 4.4 satisfies constraints (4) and (8) if and only if $z \in (0, z_3^*]$.

Theorem 4.7 (Markovian Nash equilibrium for TURF, low-intensity harvesting) *Consider the problem described in Theorem 4.4, with the addition of the constraints (4) and (8). Assume $z \in (0, z_3^*]$ and $x^\circ \in K_3(z)$, and define the strategy profiles $\psi \in \mathcal{M}(x^\circ)$ by*

$$\begin{array}{ll} \psi_1^1(x) = z\,(x_1 + \mu x_2) \wedge \beta x_1, & \psi_2^1(x) = 0 \\ \psi_1^2(x) = 0, & \psi_2^2(x) = \frac{z}{\mu}\,(x_1 + \mu x_2) \wedge \beta x_2 \end{array} \tag{39}$$

Then ψ is a Markovian Nash equilibrium at x°, in the sense of Definition 2.7. The utilities of players at equilibrium are again those described in Theorem 4.4.

4.3. **Impact of Reserve and TURF Policies.** We now compare the three scenarios: common property, reserve and TURF, for the sets of parameters that fall into the case of low-intensity harvesting. We start by analyzing how the welfare of players changes with the implementation of each of the two policies.

For concreteness, let us assume $\mu > b_2/b_1$, and set

$$\bar{z} = \min\{z_1^*, z_2^*, z_3^*\} = z_3^*, \qquad \bar{K}(z) = K_1(z) \cap K_2(z) \cap K_3(z) = K_3(z),$$

where z_i^* are defined in (27) and (38) (observe that $z_3^* \le z_1^* \le z_2^*$, for any choice of the parameters), and $K_i(z)$ in (24), (26) and (37). Then we consider a set of parameters $\Gamma_1, \Gamma_2, \alpha, \sigma, \rho$, and β such that $z = (\rho - \lambda(1-\sigma))/(2\sigma - 1) \in (0, \bar{z}]$, and $x^\circ \in \bar{K}(z)$, so that the assumptions of Theorems 3.8, 4.3 , and 4.7 are satisfied altogether. In terms of impact on the welfare of the two players, the introduction of policies is not very encouraging. If, in the case of no policy (common property case), we have

$$v_{cp}^2(x_1, x_2) = v_{cp}^1(x_1, x_2) = b_1^{1-\sigma}\ \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)},$$

then:

- Introducing the reserve strictly decreases the welfare of all players, changing it from v_{cp}^j to
$$v_{res}^2(x_1, x_2) = v_{res}^1(x_1, x_2) = \left(\frac{b_2}{\mu}\right)^{1-\sigma} \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)}$$
with $\mu > \frac{b_2}{b_1}$ implying $v_{res}^j < v_{cp}^j$.
- introducing TURF leaves the utility of the player assigned to region 1 (Player 1, according to our choice) unchanged, while it reduces the welfare of the player assigned to region 2 (Player 2) from v_{cp}^2 to
$$v_{turf}^2(x_1, x_2) = \left(\frac{b_2}{\mu}\right)^{1-\sigma} \frac{(x_1 + \mu x_2)^{1-\sigma}}{z^\sigma(1-\sigma)}$$
with $v_{turf}^2 = v_{res}^2 < v_{cp}^2$.

The introduction of policies is unsatisfactory even in terms of safeguarding the natural resource. Indeed, even if the dynamics of the two stocks $x_1(t)$ and $x_2(t)$ can be different in the three cases (a) (b) (c), at the equilibrium the total stock of the resource $x_1(t) + x_2(t)$ satisfies (16) in all cases so that

$$\frac{1}{\mu}(x_1^\circ + \mu x_2^\circ)e^{tg} \le x_1(t) + x_2(t) \le (x_1^\circ + \mu x_2^\circ)e^{tg},$$

where $g = \lambda - 2z = \frac{\lambda - 2\rho}{2\sigma - 1}$ may be positive or negative. This means that, although these policies could affect the level of the resource, they cannot affect its growth rate. In particular, when the resource is eventually exhausted in the common property case, exhaustion takes place even when the described policies are enforced.

However, since policies have an impact on the distribution of the harvesting between the two regions, they change the long-run ratio of the two stocks, biasing the ratio towards the stock of the less exploited region. Of course, the most uniform long-run distribution would follow the adoption of the TURF regime. When the effort constraint is binding, the same forces drive the long-run ratio of the two stocks, but as shown in Section 5, policies can then have a significant impact on the long-run rate of growth of the resource.

4.4. **Voracity effect.** We finally analyze the effect on the equilibrium rate of growth

$$g = \lambda - 2z = \frac{\lambda - 2\rho}{2\sigma - 1}$$

of prospective policies that increase the intrinsic reproduction rate Γ_i or reduce resource mobility α between regions.

Note that (10) implies that the eigenvalue λ is increasing with the implicit growth rates Γ_i of the two regions. The same relationship also implies that $\frac{d\lambda}{d\alpha} < 0$, so that λ increases when the resource mobility α between the two regions decreases. In particular, $\alpha > 0$ implies the tighter restriction $\lambda \in (\frac{\Gamma_1+\Gamma_2}{2}, \Gamma_2)$, and that $\lambda \to \Gamma_2$ for $\alpha \to 0$ and $\lambda \to \frac{\Gamma_1+\Gamma_2}{2}$ for $\alpha \to \infty$.

Then, as in Tornell and Lane (1999), a voracity effect characterizes our interior equilibrium when the elasticity of intertemporal substitution $\frac{1}{\sigma}$ is greater than 2 (Tornell and Lane's condition (21) on page 30), namely when the preference parameters belong to the upper shaded triangle in Figure 3(c). In this case a positive technological shock, *i.e.* an increase in the value of the eigenvalue λ, reduces the equilibrium growth rate g, as $\frac{\partial g}{\partial \lambda} = (2\sigma - 1)^{-1} < 0$. That is, each agent reacts to the the shock by increasing extraction more than proportionally and this in the end results in a fall of the post-extraction growth of the resource stocks. Moreover, the same effect takes place when mobility α between regions is reduced, as again λ increases and the overall rate of growth g decreases accordingly.

To summarize, if voracity prevails, then positive technological shocks that increase the implicit rates of growth of the two regions or reduce resource mobility between the regions lead to strategic responses that actually reduce growth.

5. Extreme Equilibria

We now consider the sets of parameters for which the internal equilibrium does not exist when a policy is implemented. We focus in particular on the case in which the agents cannot react to the creation of a reserve by adjusting their strategies as required in the internal equilibrium, because that would lead to the violation of their effort constraint. In sections 5.1 and 5.2 we construct two examples in which we are able to exhibit an extreme equilibrium and, more importantly, to show that a natural reserve policy positively affects the growth rate g of the stocks, avoiding depletion of the resource. Finally in 5.3 we broaden the discussion and draw some general conclusions.

5.1. **Logarithmic utility.** We assume $\Gamma_1 = \Gamma_2 = 2/3$, $\beta = 1$, $\alpha = 2/3$, $b_1 = b_2 = 1$ and $\rho \in (1/3, 1/2)$. As a consequence, $\lambda = 2/3$, $\mu = 1$. We also assume that players maximize the logarithmic utility (6).

(a) We first consider the case of the common resource, in which the two players are free to choose how to distribute their harvesting efforts in the two regions. Note that Remark 3.4 and Remark 3.3 apply, as long as the paths there described are feasible. That means that, if $s(t)$ denotes the overall stock of fish, namely $s(t) = x_1(t) + x_2(t)$, then a Markov equilibrium would be given by c_i^j satisfying

$$c_1^j(t) + c_2^j(t) \equiv \theta(t)\rho s(t) + (1 - \theta(t))\rho s(t) = \rho s(t), \quad \forall t \geq 0,\ j = 1, 2, \quad \theta(t) \in [0, 1], \tag{40}$$

as long as the effort constraints and the positivity of stocks are not violated. We then check that the selection $\theta = 0$ if $x_2/x_1 > 1$, $\theta = 1$ if $x_1/x_2 > 1$ and $\theta = 1/2$ if $x_2/x_1 = 1$ satisfy all constraints. Effort constraints

$$\rho s(t)\left(\frac{\theta}{x_1(t)} + \frac{1-\theta}{x_2(t)}\right) \le 1, \tag{41}$$

are satisfied, as $\rho < 1/2$ by assumption, and $s(\frac{\theta}{x_1} + \frac{1-\theta}{x_2}) \le 2$. The positivity of stocks follows from Lemma 2.6.

Now note that $\dot{s}(t) = gs(t)$, with $g = 2/3 - 2\rho < 0$ given that $\rho > 1/3$, so that the overall stock $s(t) = (x_1^\circ + x_2^\circ)e^{gt}$ is asymptotically decreasing to 0 (extinction). The corresponding utilities of players are both equal to

$$v_{cp}(x^\circ) = \int_0^{+\infty} e^{-\rho t}(\rho s(t))dt = \frac{\ln(\rho(x_1^\circ + x_2^\circ)) + 2/3 - 2\rho}{\rho}. \tag{42}$$

(b) Now we consider the case when a marine reserve is set up in region 2, forcing $c_2^1 = c_2^2 \equiv 0$. For concreteness, we suppose $x_2^\circ \ge 2x_1^\circ$. We start by showing that strategies

$$c_1^1(t) = c_1^2(t) = x_1(t) \tag{43}$$

constitute a linear Markov equilibrium. To this extent, we assume Player 2 adopts the strategy $c_1^2(t) = x_1(t)$ so that Player 1 maximizes $J^1(c^1)$ subject to

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \frac{2}{3}x_2(t) - c_1^1(t), & t \ge 0 \\ \dot{x}_2(t) = \frac{2}{3}x_1(t), & t \ge 0 \\ (x_1(0), x_2(0)) = (x_1^\circ, x_2^\circ) \end{cases} \tag{44}$$

Note that (8) is satisfied for Player 2, while it imposes $c_1^1(t) \le x_1(t)$ on Player 1. The HJB equation associated to (44) and $J^1(c^1)$ is

$$\rho v(x_1, x_2) = \left\langle \begin{pmatrix} -x_1 + \frac{2}{3}x_2 \\ \frac{2}{3}x_1 \end{pmatrix}, \nabla v(x_1, x_2) \right\rangle + \sup_{c_1^1 \in [0, x_1]} \left\{ \ln(c_1^1) - \partial_{x_1} v(x_1, x_2) c_1^1 \right\} \tag{45}$$

We show next that $c_1(t) = x_1(t)$ is the best response of Player 1. If that is true, the value function v_{res} of the problem coincides with V defined by

$$V(x^\circ) = \int_0^{+\infty} e^{-\rho t} \ln(x_1(t))dt = \int_0^{+\infty} e^{-\rho t} \ln\langle e^{tA}e_1, x^\circ\rangle dt \tag{46}$$

where $x_1(t)$ is the first component of the solution $x(t) = e^{tA}x^\circ$ to

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + \frac{2}{3}x_2(t), & t > 0 \\ \dot{x}_2(t) = \frac{2}{3}x_1(t), & t > 0 \\ (x_1(0), x_2(0)) = x^\circ \end{cases} \tag{47}$$

where $A = \begin{pmatrix} -2 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}$, that is

$$x_1(t) = \langle e^{tA}x^\circ, e_1\rangle = \langle e^{tA}e_1, x^\circ\rangle.$$

Note also that $x_2(t)/x_1(t)$ satisfies an ordinary differential equation with stationary solutions $3/2 \pm \sqrt{13}/2$ with $3/2 + \sqrt{13}/2$ an attractor. This implies, in particular, that $e^{tA}e_1$ is a vector with positive coordinates, since it is the solution to (47) with initial condition $(1, 0)$.

Next we prove V solves the HJB equation. The partial derivatives of V are given by

$$\partial_{x_1} V(x) = \int_0^{+\infty} e^{-\rho t} \frac{\langle e^{tA} e_1, e_1 \rangle}{\langle e^{tA} e_1, x \rangle} dt, \quad \partial_{x_2} V(x) = \int_0^{+\infty} e^{-\rho t} \frac{\langle e^{tA} e_1, e_2 \rangle}{\langle e^{tA} e_1, x \rangle} dt$$

Note that, since we are assuming $x_2^\circ \geq 2x_1^\circ \geq 0$, that remains true along the entire trajectory, and we have

$$0 \leq \partial_{x_1} V(x) \leq \frac{1}{x_1} \int_0^{+\infty} e^{-\rho t} \frac{\langle e^{tA} e_1, e_1 \rangle}{\langle e^{tA} e_1, e_1 + 2e_2 \rangle} dt =: \frac{1}{x_1} I < \frac{1}{x_1} \tag{48}$$

where the value of I is strictly less than 1 ($I \simeq 0.69$, with $\rho = 5/12$). Thus

$$\arg \max_{c_1^1 \in [0, x_1]} \left(\ln(c_1^1) - \partial_{x_1} V(x_1, x_2) c_1^1 \right) = x_1$$

and

$$\max_{c_1^1 \in [0, x_1]} \left(\ln(c_1^1) - \partial_{x_1} V(x_1, x_2) c_1^1 \right) = \ln(x_1) - \partial_{x_1} V(x_1, x_2) x_1.$$

Using this fact, we can see that the right hand side in (45) is equal to

$$\int_0^{+\infty} e^{-\rho t} \frac{\left\langle \left(-x_1 + \frac{2}{3} x_2\right) e_1 + \frac{2}{3} x_1 e_2, e^{tA} e_1 \right\rangle}{\langle e^{tA} e_1, x \rangle} dt + \ln(x_1) + \int_0^{+\infty} e^{-\rho t} \frac{\langle -x_1 e_1, e^{tA} e_1 \rangle}{\langle e^{tA} e_1, x \rangle} dt \tag{49}$$

$$= \int_0^{+\infty} e^{-\rho t} \frac{\langle Ax, e^{tA} e_1 \rangle}{\langle e^{tA} e_1, x \rangle} dt + \ln(x_1)$$

so that (45) is verified if and only if

$$\rho \int_0^{+\infty} e^{-\rho t} \ln \left\langle e^{tA} e_1, x \right\rangle dt = \int_0^{+\infty} e^{-\rho t} \frac{\langle Ax, e^{tA} e_1 \rangle}{\langle e^{tA} e_1, x \rangle} dt + \ln(x_1)$$

that is, if and only if

$$\int_0^{+\infty} -\frac{d}{dt} \left[e^{-\rho t} \ln \left\langle e^{tA} e_1, x \right\rangle \right] dt = \ln(x_1)$$

which is trivially satisfied. By means of a standard verification theorem one can prove that $V(x)$ defined in (46) is the value function of Player 1 when Player 2 chooses $c_1^2(t) = x_1(t)$, so that the proof that $c_1^1(t) = x_1(t)$ is the optimal response of Player 1 is complete, and $v_{res} = V$. Operating symmetrically from the standpoint of Player 2, one finally derives that (43) represents a linear symmetric Markov equilibrium. The dynamics of the system along the equilibrium is given by (47) with $x_2^\circ \geq 2x_1^\circ \geq 0$.

We can also compare the overall growth rate of the resource and the welfare of the two players, with and without the marine reserve. Note that with no reserve, (42) utilities depended on the overall stock $s(0) = x_1^\circ + x_2^\circ$, while in the presence of a reserve the result changes, for better or worse, depending on the initial distribution among the two regions. In particular, for some sets of initial conditions, welfare increases in the presence of a reserve.

To have a specific numerical illustration, consider the case $\rho = 5/12$ with $x_2^\circ = 2x_1^\circ = 2$. If the agents can access both locations, then the (selected) equilibrium path evolves for a while according to the equation

$$\begin{cases} \dot{x}_1(t) = \frac{2}{3} x_2(t) \\ \dot{x}_2(t) = -\frac{1}{6} x_1(t) - \frac{5}{6} x_2(t), \end{cases} \tag{50}$$

and upon reaching the line $x_1 = x_2$ it slides towards zero along the bisectrix of the first quadrant. In both regimes, $x_1(t) + x_2(t) = 3e^{-\frac{1}{6}t}$. With a marine reserve in region 2, the equilibrium dynamics is given instead by system (47) and the equilibrium path converges towards the positive eigenvector of matrix A, i.e. $\left(1, (3 + \sqrt{13})/2\right)'$. Note that the stock in region 1 decreases for $x_2 < 3x_1$, but eventually

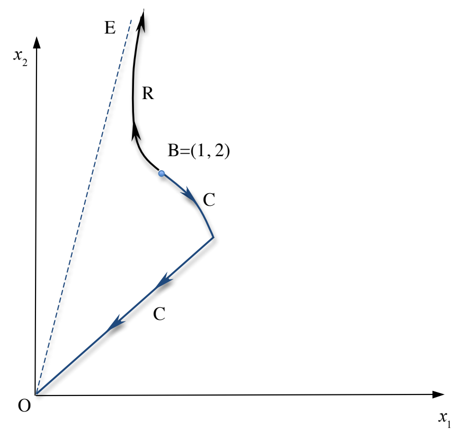

FIGURE 4. Equilibrium paths from the initial condition $B = (1; 2)$ under common property (C) and with a reserve in region 2 (R). The ray E is the positive eigenvector of matrix A.

$x_2 > 3x_1$ and both stocks grow at a positive rate. Asymptotically they grow uniformly at the rate $g = \sqrt{13}/2 - 1 > 0$. With the same choice of parameters, the asymptotic growth rate in the common resource case was $g = -1/6 < 0$. Moreover, numerically calculating the utilities of players, they increase with the introduction of the reserve, going approximately from $v_{cp}(1, 2) = -0.42$ to $v_{res}(1, 2) = 0.26$. Figure 4 illustrates the dynamics of the stocks under the two regimes, for the case $\rho = 5/12$ and initial state $B = (1, 2)$. As explained above, the trajectory of the system converges in the long run towards the origin (trajectory C) in the case of common property, and towards the direction E of the eigenvector $E = (1, (3 + \sqrt{13})/2)$ (trajectory R) in the case of the reserve. In the latter case, we have long-run positive growth.

5.2. **Power utility.** We provide here a second example for a utility of power type such as in (5). We take the same parameters as in the example of section 5.1, and show that the same argument can be used, with due changes, for values of σ sufficiently close to 1.

(a) In the case of the common resource, in equation (40) we replace ρ with $z = (\rho - (1-\sigma)\lambda)/(2\sigma - 1)$ and note that, for $\sigma > \frac{1}{2}$:

- the constraints (41) still hold, with z in place of ρ, as z satisfies $0 < z < 1/2$;
- the overall growth rate g in the free regime is negative, as

$$g = \lambda - 2z = \frac{2(\frac{1}{3} - \rho)}{2\sigma - 1} < 0; \tag{51}$$

- the objective functional can be computed explicitly and has value

$$v_{cp}(x^\circ) = \int_0^{+\infty} e^{-\rho t} \frac{(\rho s(t))^{1-\sigma}}{1-\sigma}\, dt = \frac{(z(x_1^\circ + x_2^\circ))^{1-\sigma}}{1-\sigma} \frac{1}{\rho - g(1-\sigma)}. \tag{52}$$

(b) On the other hand, in the case of the reserve:

- the value function becomes

$$v_{res}(x^\circ) = \int_0^{+\infty} e^{-\rho t} \frac{\left\langle e^{tA} e_1, x^\circ \right\rangle^{1-\sigma}}{1-\sigma} dt;$$

- the counterpart of (48), implying that the maximun of the Hamiltonian is attained on the boundary, is now $0 \le \partial_{x_1} v_{res}(x) \le {x_1}^{-\sigma} I_\sigma < {x_1}^{-\sigma}$, with

$$I_\sigma := \int_0^{+\infty} e^{-\rho t} \frac{\langle e^{tA} e_1, e_1 \rangle}{\langle e^{tA} e_1, e_1 + 2e_2 \rangle^\sigma} dt. \tag{53}$$

  One can prove that $I_\sigma < 1$ by noting that the right-hand side of (53) changes continuously with σ around 1. Since we know from the logarithmic case that (53) is satisfied for $\sigma = 1$, then it is also satisfied when σ is in a neighborhood of 1, whose size depends on the choice of ρ in $(1/3, 1/2)$;
- the proof that $v_{res}(x^\circ)$ is the solution of the HJB equation and all subsequent arguments are achieved with a straightforward modification of the proof for the logarithmic case.

5.3. **On Existence of Extreme Equilibria.** Let us summarize why both spatial property rights and reserves are at best ineffective when the coefficient of the extraction policy of the interior equilibrium is low. Note that granting secure extraction rights to Player 1 in region 1 and to Player 2 in region 2 is equivalent to setting $b_1 = 0$ in the control problem of 2 and $b_2 = 0$ in the control problem of Player 1. On the other hand, the creation of a reserve in region 1 is equivalent to setting $b_1 = 0$ for both players. If $\frac{\rho-(1-\sigma)\lambda}{2\sigma-1} \le \min\{z_1^*, z_2^*\}$, then each player reacts to these new regulations by adapting the linear policy that characterizes our interior Markov equilibrium to the new set of parameters. The growth rate of the resource is unaffected by this change of behavior and the welfare of each player is at best unchanged. In the specific example above, we have $z_1^* = z_2^* = \frac{1}{3}$ and a logarithmic utility (i.e., $1 - \sigma = 0$), so we find policy ineffectiveness if $\rho \le \frac{1}{3}$.

However, we have also shown that if $\frac{\rho-(1-\sigma)\lambda}{2\sigma-1} > \min\{z_1^*, z_2^*\}$ but the interior equilibrium still exists under the common property or TURF regimes, then creating a natural reserve in the region with the maximum z^* can increase the growth rate of the resource and player welfare. Indeed an extreme equilibrium in which the effort constraint is binding can exist in this case. Although a complete analysis of the existence conditions of this extreme equilibrium is not given in this paper, our example is sufficient to show that the set parameters for which it exists is nonempty. More generally, even a TURF management system can be somewhat effective if, when instituted, the effort constraint of at least one player is binding.

6. Extensions of the model

A series of extensions of the model are possible. Some are straightforward, while some others would require a detailed discussion which is beyond the purpose of the present work. We here sketch out some of them.

6.1. **Heterogeneous catchability parameters.** Consider the case when the catchability parameter β, which depends on geographical features, is different in the two regions; namely β_j in region j, $j = 1, 2$. Then (8) and (8) become

$$c_i^j(t) = \beta_j E_i^j(t) x_i(t), \text{ and } \frac{c_1^j(t)}{\beta_1 x_1(t)} + \frac{c_2^j(t)}{\beta_2 x_2(t)} \le 1.$$

Theorems 3.8, 4.3 and 4.7 can be proved with insignificant changes, namely:

- in $K_j(z)$ and z_j^*, take $\beta = \beta_j$; set

$$K_3(z) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 : \frac{\mu z}{\beta_1 - z} x_2 \le x_1 \le \frac{\mu(\beta_2 - z)}{z} x_2 \right\},$$

  and replace $z_3^*(\beta)$ with $z_3^*(\beta_1) \wedge (\beta_1/2) \wedge (\beta_2/2)$;
- in the equilibrium strategies, replace βx_j with $\beta_j x_j$.

6.2. **Heterogeneous preferences.** In the model of Section 2 the agents are identical in terms of preference parameters ρ and σ. Indeed the results can be generalized to the case where the parameters are different for the two agents: ρ^j and σ^j for Player j. The arguments for the interior solution case are the same as in Theorems 3.1, 4.1 and 4.4, with the difference that instead of having a unique $z = \frac{\rho-\lambda(1-\sigma)}{2\sigma-1}$ we have two player-specific coefficients z^1 and z^2 solving

$$\begin{cases} z^1 = \frac{\rho^1-\left(\lambda-z^2\right)\left(1-\sigma^1\right)}{\sigma^1} \\ z^2 = \frac{\rho^2-\left(\lambda-z^1\right)\left(1-\sigma^2\right)}{\sigma^2} \end{cases}$$

that is

$$z^1 = \frac{\rho^2+\sigma^2(\rho^1-\rho^2)-\lambda(1-\sigma^1)}{\sigma^1+\sigma^2-1}, \qquad z^2 = \frac{\rho^1+\sigma^1(\rho^2-\rho^1)-\lambda(1-\sigma^2)}{\sigma^1+\sigma^2-1}.$$

The counterpart of Theorem 3.8 for the case $\mu > b_1/b_2$ reads, for example, as follows: if $z^1 \in (0, z_1^*]$ (with z_1^* defined in Proposition 3.7), $z^2 \in (0, z_1^*]$ and $x^\circ \in K_1(z_1) \cap K_1(z_2)$, then a Markovian equilibrium is given by

$$\begin{array}{ll} c_1^1(t) = z^1\left(x_1(t) + \mu x_2(t)\right) \wedge \beta x_1, & c_2^1(t) \equiv 0 \\ c_1^2(t) = z^2\left(x_1(t) + \mu x_2(t)\right) \wedge \beta x_1, & c_2^2(t) \equiv 0. \end{array} \tag{54}$$

The results of Theorems 4.1 and 4.4 extend similarly. Thus, overall conclusions do not change in the case of low harvesting: for sets of parameters where the assumptions of Theorems 3.1, 4.1 and 4.4 hold at the same time, the introduction of a reserve/TURF does not affect the overall growth rate of the resource, while it strictly decreases the welfare of at least one player.

6.3. **Preferences for resource conservation.** In the specifications of the model given in Section 2, the utility only depends on the harvesting flow, and when agents choose to not completely deplete the resource it is only to be able to continue exploiting it in the future. Although this is a common approach, one may think of alternative specifications in which agents (for instance local authorities making decisions about territories under their jurisdiction) have a direct interest in conservation. That translates into a stock-dependent utility function where the stocks positively affect the payoffs. In the simplest case of a separable function in which utility depends linearly on the stocks, we can assume:

$$J^j(c^j) = \int_0^{+\infty} e^{-\rho t}\left[\left(d_1^j x_1(t) + d_2^j x_2(t)\right) + \frac{\left(b_1 c_1^j(t) + b_2 c_2^j(t)\right)^{1-\sigma}}{1-\sigma}\right] dt \tag{55}$$

where d_i^j is the weight that Player j attributes to safeguarding the resource in region i. To ensure finite overall utilities, we assume $\rho > \max\{\Gamma_1, \Gamma_2\}$. In the common property case, in suitable sets of data and for a large enough initial datum, there exists an internal equilibrium in the spirit of Definition 2.7. The strategy of players at equilibrium is the following: Player 1 decides to fish only in the region i with the smallest coefficient

$$a_i^1 := \frac{d_i^1(\rho+\alpha-\Gamma_{3-i}) + \alpha d_{3-i}^1}{(\rho-\Gamma_1+\alpha)(\rho-\Gamma_2+\alpha)-\alpha^2} b_{3-i}.$$

That is, Player 1 will fish in region 1 if and only if $\left(d_1^1(\rho+\alpha-\Gamma_2)+\alpha d_2^1\right) b_2 < \left(d_2^1(\rho+\alpha-\Gamma_1)+\alpha d_1^1\right) b_1$, and will fish in region 2 if the inequality goes the other way. Denote by $\bar{i}$ the chosen region. The player's consumption will then be equal to

$$c_{\bar{i}}^1(t) = \left(a_{\bar{i}}^1 \frac{b_{\bar{i}}}{b_{3-\bar{i}}}\right)^{-1/\sigma}.$$

A symmetric statement holds for Player 2. Analysis of the marine reserve or TURF cases yields similar results. When an internal solution is available, one may observe the following:

(i) Harvesting flows decrease with the weights d_i^j, i.e. with the importance the agents attribute to safeguarding the resource. The choice of the region for harvesting depends now on the coefficients d_i^j as well, and not only on the productivities Γ_i.
(ii) When the introduction of policies changes the decisions of players, the level of the resource increases but, as in our baseline case, the growth rate is not affected.
(iii) We cannot obtain the results in Theorem 3.8 for the baseline model as the limit for $d_i^j \to 0$ of the modified model because the solution of the latter is only feasible for initial values which are sufficiently large given the data of the problem, but such a threshold becomes infinite when $d_i^j \to 0$.

6.4. **The n-region problem.** A fourth, more substantial, generalization of the model would take into account an n-region set-up. As a first step one could consider the case of n regions with intrinsic reproduction rates $\Gamma_1, \dots, \Gamma_n$ and a common diffusion coefficient $\frac{\alpha}{n-1}$ between any two locations. In this case the counterpart of the matrix M in (9) is

$$M_n = \begin{pmatrix} \Gamma_1 - \alpha & \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \dots & \frac{\alpha}{n-1} \\ \frac{\alpha}{n-1} & \Gamma_2 - \alpha & \frac{\alpha}{n-1} & \dots & \frac{\alpha}{n-1} \\ \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \Gamma_3 - \alpha & \dots & \frac{\alpha}{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \dots & \Gamma_n - \alpha \end{pmatrix}.$$

Let λ be the greatest eigenvalue of M_n. As in the two-region model, λ is a "weighted" average of the different Γ_i, representing the overall growth rate of the resource, based on geographical features of those regions. Also in this case the value of λ increases with increasing Γ_i, and with decreasing α. It is an interesting question, but beyond the scope of the present work, as to whether the results of Theorems 3.8, 4.3 and 4.7 may be replicated also in this n-region case. Finally we mention that a further interesting generalization would be to consider different diffusion coefficients among regions, specifying spatial heterogeneities even further.

6.5. **More than two agents.** Even within the limits of a 2-region setting, increasing the number of agents with access to the resources would deepen the understanding of the effect of competition on conservation of stocks. With N agents, and with $\sigma \neq 1$, the coefficient of the policy function at the interior equilibrium becomes $z_N = \frac{\rho-(1-\sigma)\lambda}{N(\sigma-1)+1} > 0$, implying that the number of agents affects the extraction policy of each one. (Only for a logarithmic utility function, which is morally the case $\sigma = 1$, that coefficient coincides with the discount rate ρ, independently of the number of agents, causing the total extraction to increase linearly with N.) With $N_1 \leq N$ agents extracting from region 1 and $N - N_1$ extracting from region 2, at the equilibrium of the relaxed problem the stocks evolve according to the equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1 - \alpha - N_1 z_N & \alpha - N_1 z_N \mu \\ \alpha - \frac{(N-N_1)z_N}{\mu} & \Gamma_2 - \alpha - (N-N_1)z_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{56}$$

Note that increased competition guarantees existence of the interior equilibrium of the constrained problem. Indeed, when N is sufficiently large, z_N becomes small enough to satisfy the effort constraints. Moreover, when the chosen parameters satisfy

$$\frac{\alpha}{\mu} > \frac{\rho - (1-\sigma)\lambda}{\sigma - 1},$$

then

$$\alpha\mu \geq \frac{\alpha}{\mu} > \frac{\rho - (1-\sigma)\lambda}{(\sigma-1) + \frac{1}{N}} \geq \frac{\rho - (1-\sigma)\lambda}{(\sigma-1) + \frac{1}{N}} \max\left\{\frac{N_1}{N}; \frac{N-N_1}{N}\right\},$$

so that the positivity constraint is also satisfied.

At the interior equilibrium, the growth rate of the resource is given by $g = \lambda - Nz_N$, and so increasing competition would increase the growth rate of the resource if and only if $\sigma < 1$ (see, Tornell and Lane, 1999).

Finally, if we introduce congestion externalities by assuming that iceberg costs $1-b_1$ and $1-b_2$ increase with the number of agents extracting from the corresponding region, this creates incentives for agents to switch to the less crowded region, and could be used to endogenize the spatial distribution of the N agents.

7. Conclusions

In this paper we have developed a continuous time model in which two players compete to exploit a resource which can move between two regions. The model we provide is analytically tractable, and able to capture the difficulties of designing an efficient system of spatial property rights in the context of spatially distributed resources.

We compare the behaviors of the players in an initial common-property case (where both can decide where and how much to harvest) with their choices in two policy-constrained cases: the first where the regulator can establish a natural reserve (namely, where harvesting in one of the two regions is forbidden) and the second where each player has exclusive exploitation rights for one of the two locations. We show that the policies are completely ineffective (and detrimental to player utilities) when conditions lead the players to choose a low-intensity exploitation of the resource, whereas the policies can be useful for safeguarding the resource, and benefit player utilities, when agents are induced to fully use their total effort capacity.

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Appendix A. Proofs

This Appendix contains the proofs of all theorems stated in the previous sections.

Proof of Lemma 2.2 Consider the trajectory $x^\psi(t)$ starting at $x^\circ \in S$ and assume that there exists a $\bar{t} \geq 0$ such that $x^\psi(\bar{t}) \in S_0 = \{x \in \mathbb{R}^2_+ : \langle x, \eta \rangle = 0\}$. Consider also the function $x(t) = e^{(t-\bar{t})M} x^\psi(\bar{t})$ for all $t \geq \bar{t}$. We want to show that, for all $t \geq \bar{t}$: (*i*) $x(t) \in S_0$; (*ii*) $x^\psi(t) \equiv x(t)$. Since $e^{s\lambda}$ is an eigenvalue of e^{sM} of eigenvector η, the first fact follows from

$$\langle x(t), \eta \rangle = \left\langle x^\psi(\bar{t}), e^{(t-\bar{t})M}\eta \right\rangle = e^{(t-\bar{t})\lambda} \left\langle x^\psi(\bar{t}), \eta \right\rangle = 0.$$

To prove the second we notice that, for all $t \geq \bar{t}$, $x(t) \in S_0$ implies $\psi_i^j(x(t)) = 0$ for all i, j, and hence $x(t)$ solves the differential system (12), with initial condition $x(\bar{t}) = x^\psi(\bar{t})$. Since that solution is unique and x^ψ is also a solution, then $x(t) = x^\psi(t)$ for all $t \geq \bar{t}$. The statement of the lemma then follows. □

Proof of Lemma 2.6 Effort constraints imply that $\psi_i^j(x) \leq \beta x_i$, so that $x^\psi(t)$ satisfies for all t

$$\dot{x}_i^\psi(t) \geq (\Gamma_i - \alpha - 2\beta)x_i^\psi(t) + \alpha x_{(3-i)j}^\psi(t), \quad i = 1, 2.$$

Then it is sufficient to observe (see e.g. Farina and Rinaldi, 2000, Chapter 2) that

$$\dot{x}(t) = \begin{pmatrix} \Gamma_1 - \alpha - 2\beta & \alpha \\ \alpha & \Gamma_2 - \alpha - 2\beta \end{pmatrix} x(t), \qquad x(0) = x^\circ$$

is a positive system, that is, its trajectories starting in the positive orthant remain there at all times, as the matrix of the system is a Metzler matrix. □

Proof of Theorem 3.1 We analyze the case $\mu > \frac{b_1}{b_2}$; the opposite case can be treated similarly.
Step 1: Solution of the Hamilton-Jacobi-Bellman equation for Player 1. We assume Player 2 fishes only in region 1, using a feedback strategy $\psi^2 \equiv (\psi_1^2, \psi_2^2)$ defined by $\psi_1^2(x) = w \langle \eta, x \rangle$ with $w \in \mathbb{R}^+$, and $\psi_2^2(x) = 0$, for $x \in S$. Then, consistently with Definition 2.3, Player 1 has to choose $c^1 \in \mathcal{C}^{\psi^2}(x^\circ)$ so as to maximize $J^1(c^1)$ given by (5), and subject to

$$\dot{x}(t) = Mx(t) - w \langle \eta, x(t) \rangle e_1 - c^1(t), \quad x(0) = x.$$

For $(x, p, c) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ we set

$$h(x, p, c) = \langle p, Mx \rangle - w \langle \eta, x \rangle \langle p, e_1 \rangle + \frac{\left(b_1 c_1^1 + b_2 c_2^1\right)^{1-\sigma}}{1 - \sigma} - \left\langle p, c^1 \right\rangle$$

so that the Hamiltonian function is given by

$$H(x, p) = \sup_{c_1^1, c_2^1 \geq 0} h(x, p, c) = \langle p, Mx \rangle - w \langle \eta, x \rangle \langle p, e_1 \rangle + \frac{\sigma}{1 - \sigma} \min \left\{ \frac{p_1}{b_1}, \frac{p_2}{b_2} \right\}^{1-1/\sigma}$$

where the supremum is attained on the boundary, either at

$$c_1^{1*} = \frac{1}{b_1} \left(\frac{p_1}{b_1} \right)^{-\frac{1}{\sigma}}, \quad c_2^{1*} = 0 \tag{57}$$

or at

$$c_1^{1*} = 0, \quad c_2^{1*} = \frac{1}{b_2} \left(\frac{p_2}{b_2} \right)^{-\frac{1}{\sigma}}.$$

It is then easy to check that the HJB equation $\rho v(x) = H(x, \nabla v(x))$ associated to the problem has a solution of type $v(x) = \frac{1}{1-\sigma} \left(\beta \langle \eta, x \rangle\right)^{1-\sigma}$. Indeed, $\nabla v(x) = \beta^{1-\sigma} \langle \eta, x \rangle^{-\sigma} \eta$ and $M\eta = \lambda\eta$ imply

$$\langle \nabla v(x), Mx \rangle = \langle M \nabla v(x), x \rangle = \beta^{1-\sigma} \langle \eta, x \rangle^{-\sigma} \langle M\eta, x \rangle = \lambda \beta^{1-\sigma} \langle \eta, x \rangle^{1-\sigma}.$$

Moreover, since $\mu > \frac{b_1}{b_2}$, one has

$$\min \left\{ \frac{\partial_1 v(x)}{b_1}, \frac{\partial_2 v(x)}{b_2} \right\} = \beta^{1-\sigma} \langle \eta, x \rangle^{-\sigma} \min \left\{ \frac{1}{b_1}, \frac{\mu}{b_2} \right\} = \frac{1}{b_1} \beta^{1-\sigma} \langle \eta, x \rangle^{-\sigma}.$$

Hence the HJB equation is satisfied by

$$v(x) = \frac{1}{1-\sigma}\left(\beta\left\langle \eta, x\right\rangle\right)^{1-\sigma}, \text{ with } \beta = \left(\frac{\sigma}{\rho - (\lambda - w)(1-\sigma)}\right)^{\frac{\sigma}{1-\sigma}} b_1. \tag{58}$$

Step 2: Nash equilibrium. From (57), the optimal fishing strategy for Player 1 is

$$c_1^{1*}(t) = \frac{\rho - (\lambda - w)(1-\sigma)}{\sigma}\left\langle \eta, x(t)\right\rangle, \quad c_2^{1*}(t) = 0.$$

We may repeat the argument from the standpoint of Player 2, assuming Player 1 has a strategy of type $\psi_1^1(x) = u\left\langle \eta, x\right\rangle$ for some $u \in \mathbb{R}^+$ and $\psi_2^1(x) = 0$, for $x \in S$, deriving

$$c_1^{2*}(t) = \frac{\rho - (\lambda - u)(1-\sigma)}{\sigma}\left\langle \eta, x(t)\right\rangle, \quad c_2^{2*}(t) = 0.$$

Then $\left(c^{1*}, c^{2*}\right)$ is a Nash equilibrium if and only if $c^1 = c^{1*}$ and $c^2 = c^{2*}$, that is

$$\begin{cases} w = \frac{\rho - (\lambda - u)(1-\sigma)}{\sigma} \\ u = \frac{\rho - (\lambda - w)(1-\sigma)}{\sigma} \end{cases}$$

which implies

$$w = u = \frac{\rho - \lambda(1-\sigma)}{2\sigma - 1} = z, \quad \beta = z^{\frac{\sigma}{\sigma - 1}}$$

and therefore, (14) and (15). Then equation (2) becomes, at equilibrium,

$$x'(t) = \begin{pmatrix} \Gamma_1 - \alpha & \alpha \\ \alpha & \Gamma_2 - \alpha \end{pmatrix} x(t) - 2z \begin{pmatrix} 1 & \mu \\ 0 & 0 \end{pmatrix} x(t). \tag{59}$$

As a consequence (recall that λ is the eigenvalue of M associated to η),

$$\frac{d}{dt}\left\langle x(t), \eta\right\rangle = (\lambda - 2z)\left\langle x(t), \eta\right\rangle$$

so that

$$\left\langle x(t), \eta\right\rangle = e^{(\lambda - 2z)t}\left\langle x^\circ, \eta\right\rangle \geq 0, \ \forall t \geq 0. \tag{60}$$

and so the trajectory remains in S and satisfies (16).

Step 3: Verification Theorem. In order to prove that the solution of the HJB equation is in fact the value function of the problem for the first player, one has to prove that

$$v(x^\circ) = V(x) \equiv \sup_{c^1 \in \mathcal{C}^{\psi^2}(x^\circ)} J(c^1).$$

As v solves the HJB equation, we have

$$v(x^\circ) - v(x(T))e^{-\rho T} = -\int_0^T \frac{d}{dt} e^{-\rho t} v(x(t))dt = \int_0^T e^{-\rho t}\big(H(x(t), \nabla v(x(t))) - \left\langle \nabla v(x), \dot{x}(t)\right\rangle\big)dt.$$

For any admissible control $c^1 \in \mathcal{C}^{\psi^2}(x^\circ)$. We may subtract $J_T(c_1) = \int_0^T e^{-\rho t}\frac{1}{1-\sigma}(b_1 c_1^1(t) + b_2 c_2^1(t))^{1-\sigma}$ from both sides of the above equality, so to obtain the fundamental identity

$$v(x^\circ) - J_T(c_1) = \int_0^T e^{-\rho t}\Big[H\big(x(t), \nabla v(x(t))\big) - h\big(x(t), \nabla v(x(t)), c^1(t)\big)\Big]dt + v(x(T))e^{-\rho T}. \tag{61}$$

We study separately the two cases $\sigma \in (0,1)$ and $\sigma > 1$.

In the case $\sigma \in (0,1)$, v given by (58) is positive, and the argument of the integral in (61) is also positive by definition of h and H. Then the whole right hand side in (61) is positive, implying

$$v(x^\circ) \geq J_T(c_1), \text{ for all } c^1 \in \mathcal{C}^{\psi^2}(x^\circ)$$

and, taking the supremum for $c^1 \in \mathcal{C}^{\psi^2}(x^\circ)$,

$$v(x^\circ) \geq V(x^\circ).$$

Note now that that $\lim_{t\to+\infty} e^{-\rho t} v(x^*(t)) = 0$. Indeed, along the equilibrium trajectories (60) holds, and then $e^{-\rho t} v(x^*(t))$ decreases exponentially to zero with rate

$$-\rho + (1-\sigma)(\lambda - 2z) = -z < 0. \tag{62}$$

Thus, evaluating the fundamental identity at c^{1*} and taking the limits as $T \to \infty$, we derive also

$$v(x^\circ) = J_1(c^{1*}) \leq V(x^\circ).$$

Then v is Player 1's value function, and $\psi^1(x)$ is the optimal response to Player 2's strategy $\psi^2(x)$.

The previous argument does not apply to the case $\sigma > 1$, as the candidate value function v is negative (along with all utilities). Nonetheless one can prove that $\lim_{t\to+\infty} e^{-\rho t} v(x(t)) = 0$ along any trajectory x of the system. To this extent, note that when Player 2 fishes only in region 1 with intensity $z\langle \eta, x\rangle$, the growth rate of the quantity $\langle x(t), \eta\rangle$ is at most $\lambda - z$ (corresponding to Player 1 not fishing at all). More precisely, arguing as for (60), one has

$$|\langle x(t), \eta\rangle| \leq |\langle x^\circ, \eta\rangle| e^{(\lambda - z)t}, \quad \forall c^1 \in \mathcal{C}^{\psi^2}(x^\circ) \text{ and } x(t) = x(t; c^1).$$

Now, since by (62)

$$-\rho + (1-\sigma)(\lambda - z) = -z + (1-\sigma)z = -\sigma z < 0, \tag{63}$$

then[3]

$$\left|e^{-\rho t} v(x(t))\right| = \frac{1}{|1-\sigma|} e^{-\rho t} \left|\langle \eta, x(t)\rangle\right|^{1-\sigma} \leq \frac{\left|\langle \eta, x^\circ\rangle\right|^{1-\sigma}}{|1-\sigma|} e^{(-\rho + (\lambda - z)(1-\sigma))t} \xrightarrow{t\to\infty} 0.$$

As a consequence, taking the limits as T tends to $+\infty$ in (61), we obtain that

$$v(x^\circ) - J(c_1) = \int_0^{+\infty} e^{-\rho t} \Big[H\big(x(t), \nabla v(x(t))\big) - h\big(x(t), \nabla v(x(t)), c^1(t)\big) \Big] dt$$

holds along all trajectories. Finally, since the right hand side of the above identity is positive

$$v(x^\circ) \geq J(c_1), \text{ for all } c^1 \in \mathcal{C}^{\psi^2}(x^\circ),$$

and the conclusion follows as in the case $\sigma \in (0,1)$.

We may then repeat the argument from the standpoint of Player 2, reacting to the choice ψ^1 of Player 1, and reach the conclusion that (ψ^1, ψ^2) is a Markovian Nash equilibrium in the sense of Definition 2.3. □

Proof of Lemma 3.5 The proof is straightforward: one verifies that $\bar{\eta}$ and $(u_1, 1)$ are eigenvectors of the matrix B, associated to eigenvalues $\bar{\lambda}$ and λ respectively, making use of the identities

$$\Gamma_1 - \alpha + \alpha\mu = \lambda, \qquad \alpha\lambda + (\Gamma_2 - \alpha)\mu = \lambda\mu, \tag{64}$$

obtained from $M\eta = \lambda\eta$. □

Proof of Proposition 3.6. We prove the statements for the case $\mu > \frac{b_2}{b_1}$; the proof for the other case is similar. We set $u^\circ = x_1^\circ / x_2^\circ$, $u(t) = x_1(t)/x_2(t)$, and see that u satisfies

$$u'(t) = -\alpha u^2(t) - (\Gamma_2 - \Gamma_1 + 2z)\, u(t) + (\alpha - 2z\mu)\,. \tag{65}$$

whose stationary solutions u_1, u_2 of (65) are $u_1 = \frac{1}{\mu} - \frac{2z}{\alpha}$ and $u_2 = -\mu$. Both statements are then derived from the convergence of the trajectory $u(t)$ of (65) to u_1 when $u^\circ > -\mu$ and, a fortiori, when $x^\circ \in \mathbb{R}^2_2$. □

[3] A similar argument holds for logarithmic utility. Indeed in that case the growth of the term $v(x(t))$ (described in Remark 3.4) is at most linear in t so that $\left|e^{-\rho t} v(x(t))\right|$ converges to 0 along any admissible trajectory.

Proof of Proposition 3.7. Assume $\mu > b_2/b_1$. By Lemma 2.6 and Proposition 3.6, it is not restrictive to assume $z \in (0, \alpha/(2\mu)]$, where positivity constraints are satisfied. Elementary-but-tedious calculations show that $z_1^* \in [0, \alpha/(2\mu)]$ and that $(u_1(z), 1) \in K_1(z)$ if and only if $z \in [0, z_1^*]$. Now fix $z \in [0, z_1^*]$, and $x^\circ = (x_1^\circ, x_2^\circ) \in K_1(z)$. Then $u^\circ = x_1^\circ/x_2^\circ > \mu z/(\beta - z)$. Since the trajectory of (65) starting at u° converges to the value $u_1(z) > \mu z/(\beta - z)$ then the trajectory $u(t)$ satisfies $u(t) > \mu z/(\beta - z)$ for all $t \geq 0$, which implies (23) and hence (8). The case $\mu < b_2/b_1$ can be similarly discussed; we omit the proof for brevity. □

Proof of Theorem 3.8 The proof follows that of Theorem 3.1. We list changes below for the case $\mu > b_2/b_1$.

We assume that Player 2 fishes only in region 1, with intensity $z\,(x_1 + \mu x_2) \wedge \beta x_2$. Thus Player 1 has to choose c^1 so as to maximize $J^1(c^1)$ given by (5), and subject to

$$\dot{x}(t) = Mx(t) - z\,(\langle \eta, x(t)\rangle \wedge \beta x_2)\, e_1 - c^1(t), \quad x(0) = x.$$

Set $h(x,p,c) = \frac{1}{1-\sigma}(b_1 c_1^1 + b_2 c_2^1)^{1-\sigma} - \langle p, M - z\,(\langle \eta, x\rangle \wedge \beta x_2)\, e_1 - c^1$. Then

$$H(x,p) \sup_{c_1^1, c_2^1 \geq 0} h(x,p,c) = \langle p, Mx\rangle - (z\,\langle \eta, x\rangle \wedge \beta x_2)\,\langle p, e_1\rangle + \frac{\sigma}{1-\sigma} \min\left\{\frac{p_1}{b_1}, \frac{p_2}{b_2}\right\}^{1-1/\sigma} \tag{66}$$

and $\rho v(x) = H(x, \nabla v(x))$ is the HJB equation of the problem.

One cas easily verify that the function $v(x) = (b_2/\mu)^{1-\sigma} \frac{(x_1+\mu x_2)^{1-\sigma}}{z^\sigma(1-\rho)}$ is a solution of the HJB equation for all $x \in K_1(z)$ and the trajectory $x(t)$ remains in the cone $K_1(z)$ at all times by Proposition 3.7. As a consequence, the identity (61) holds also in this case, for any choice of the controls c^1 in $\mathcal{C}^{\psi^2}(x^\circ)$ (even for those that drive the associated trajectories out of cone $K_1(z)$) so that v is the value function of the problem and $\psi^1(x)$ the optimal response of Player 1 to the strategy $\psi^2(x)$ of Player 2, and (ψ^1, ψ^2) is a Markovian Nash equilibrium in the sense of Definition 2.7. □

Proof of Theorem 4.4 We proceed as in the proof of Theorem 3.1. Assume that the strategy of Player 2 is of type $c_2(t) = w\langle \eta, x(t)\rangle$ with $w > 0$ given. Then Player 1 has to choose c_1 so as to maximize $J^1(c^1)$ and subject to

$$\dot{x}(t) = Mx(t) - w\,\langle \eta, x(t)\rangle\, e_2 - c_1(t) e_1, \quad x(0) = x^\circ.$$

The Hamiltonian function associated to the problem, for $(x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, is

$$\begin{aligned} H(x,p) &= \langle p, Mx\rangle - w\,\langle \eta, x\rangle\,\langle p, e_2\rangle + \sup_{c_1 \geq 0}\left\{\frac{(b_1 c_1)^{1-\sigma}}{1-\sigma} - c_1 p_1\right\} \\ &= \langle p, Mx\rangle - w\,\langle \eta, x\rangle\, p_2 + \frac{\sigma}{1-\sigma}\left(\frac{p_1}{b_1}\right)^{1-\frac{1}{\sigma}} \end{aligned}$$

with supremum attained at

$$c_1 = \frac{1}{b_1}\left(\frac{p_1}{b_1}\right)^{-\frac{1}{\sigma}}. \tag{67}$$

The HJB equation $\rho v(x) = H(x, \nabla v(x))$ has a solution of type $v(x) = \frac{1}{1-\sigma}\,(\beta\,\langle \eta, x\rangle)^{1-\sigma}$, with

$$\beta = b_1 \left(\frac{\rho - (\lambda - \mu w)\,(1-\sigma)}{\sigma}\right)^{\frac{\sigma}{\sigma-1}},$$

as is easy to check directly. From (67), the optimal fishing strategy for Player 1 is

$$c_1^* = \frac{\rho - (\lambda - w\mu)(1-\sigma)}{\sigma}\,\langle \eta, x\rangle\,.$$

We may repeat the argument from the standpoint of Player 2, assuming Player 1 has a strategy of type $c_1(t) = u\langle \eta, x(t)\rangle$ for some $u \in \mathbb{R}^+$, deriving a solution of the HJB equation of type $v(x) = (1-\sigma)^{-1}(\gamma\langle \eta, x\rangle)^{1-\sigma}$, with

$$\gamma = \frac{b_2}{\mu}\left(\frac{\rho-(1-\sigma)(\lambda-u)}{\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

and the optimal strategy

$$c_2^* = \frac{\rho-(\lambda-u)(1-\sigma)}{\sigma\mu}\langle \eta, x\rangle .$$

Then (c_1^*, c_2^*) is a Nash equilibrium if and only if $c_1 = c_1^*$ and $c_2 = c_2^*$, that is

$$\begin{cases} w = \frac{\rho-(\lambda-u)(1-\sigma)}{\sigma\mu} \\ u = \frac{\rho-(\lambda-\mu w)(1-\sigma)}{\sigma} \end{cases}$$

which implies

$$\mu w = u = \frac{\rho-\lambda(1-\sigma)}{2\sigma-1} = z, \quad \beta = b_1\, z^{\frac{\sigma}{\sigma-1}}, \quad \gamma = \frac{b_2}{\mu}\, z^{\frac{\sigma}{\sigma-1}}$$

and hence, (31) and (32). It is straightforward to show that the defined strategies satisfy part (i) of Definition 2.5.

If we compute $\frac{d}{dt}\left\langle x^{\psi}(t), \eta\right\rangle$ we get $\left\langle Mx^{\psi}(t), \eta\right\rangle - \frac{z}{\mu}\left\langle x^{\psi}(t), \eta\right\rangle\langle(\mu,1),(1,\mu)\rangle = \left\langle x^{\psi}(t), \eta\right\rangle(\lambda-2z)$ and then we get (33). The verification part of the proof is as in Theorem 3.1.

□

Proof of Theorem 4.7 The proof follows by adapting the proof of Theorem 4.4, exactly as the proof of Theorem 3.8 was obtained from the proof of Theorem 3.1. In this case, the solution of the HJB equation will only be defined on $K_3(z)$.

□

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