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The Kumaraswamy skew-normal distribution

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Abstract

We propose a new generalization of the skew-normal distribution (Azzalini, 1985) referred to as the Kumaraswamy skew-normal. The new distribution is computationally more tractable than the Beta skew-normal distribution (Mameli and Musio, 2013) with which it shares some properties.

Keywords: Skew-normal distribution, Beta skew-normal distribution, Kumaraswamy distribution, Kumaraswamy skew-normal distribution.

1. Introduction

Some recent developments in distribution theory have proposed new techniques for building distributions. Among these, the methods used to construct the Beta generalized ($Beta - F$) (Jones, 2004) and the Kumaraswamy generalized ($Kw{-}F$) (Cordeiro and de Castro, 2011) class of distributions have received a lot of attention. The first work concerning the Beta-generated family was proposed by Eugene et al. (2002), who defined and analysed the Beta-normal distribution. Further, Jones (2004) formalized the definition of the Beta-generated family. Its work has inspired many researchers and has fuelled an enormous literature regarding this family of distributions; see for example Gupta and Nadarajah (2005), Pescim et al. (2010), Mameli and Musio (2013). Recently, following the idea of the class of Beta-generated distributions (Jones, 2004), Cordeiro and de Castro (2011) proposed a new family of generalized distribu-

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tions, called Kumaraswamy generalized family, by means of the Kumaraswamy distribution (Kumaraswamy, 1980; Jones, 2009). The maximum likelihood estimation for the family $Kw - F$ distribution results simpler than the estimation in the $Beta - F$ family. Motivated by these facts, we define in this paper a new generalization of the skew-normal based on the Kumaraswamy generalized family, which is more tractable of the Beta skew-normal (BSN) introduced by Mameli and Musio (2013). The resulting distribution, which will be called the Kumaraswamy skew-normal ($KwSN$), could be considered a valid alternative to the BSN distribution with which it shares some similar properties. Moreover, for special values of the parameters the $KwSN$ distribution is related to the Beta skew-normal one. The $KwSN$ distribution is always unimodal, unlike the Beta skew-normal which can be either unimodal or bimodal. The $KwSN$ model shows more flexibility than the SN one. Furthermore, under the null hypothesis of normality the $KwSN$ distribution, as the BSN one, is not identifiable. However, due to the tractability of the $KwSN$ density, all the possible sets of parameters for which this density reduces to the normal one can be established by exploiting the Lambert W function; see Jeffrey et al. (1998). The rest of the paper organizes as follows. Section 2 defines the $KwSN$ distribution and presents some properties of the new distribution. Section 3 investigates maximum likelihood estimation and analyses a data set of Australian athletes measurements. Finally, concluding remarks are given in Section 4.

2. The new model

In this section we first define the Kumaraswamy skew-normal distribution and then we present some of its properties.

2.1. Definition and simple properties

For a given distribution function $F(x)$, with associated density function (pdf) $f(x)$, Cordeiro and de Castro (2011) represented the cumulative distribution function of the Kumaraswamy-F distribution as

$$G^K_{F(x)}(x; a, b) = 1 - (1 - F(x)^a)^b, \text{ with } x \in \mathbb{R}. \tag{1}$$

The density function correspondent to (1) is

$$g^K_{F(x)}(x; a, b) = abf(x)F(x)^{a-1}(1 - F(x)^a)^{b-1}, \text{ with } x \in \mathbb{R}. \quad (2)$$

The family of these distributions will be indicated by $Kw - F(a, b)$. Replacing the distribution function of the skew-normal $\Phi(x; \lambda)$ in (2), we obtain a new distribution, which will be called the Kumaraswamy skew-normal distribution ($KwSN(\lambda, a, b)$), whose density function is

$$g^K_{\Phi(x;\lambda)}(x; \lambda, a, b) = ab\phi(x; \lambda)(\Phi(x; \lambda))^{a-1}(1 - \Phi(x; \lambda)^a)^{b-1}, \text{ with } x \in \mathbb{R}. \quad (3)$$

This class can be generalized by including a location parameter μ and a scale parameter $\sigma > 0$. Thus if $X \sim KwSN(\lambda, a, b)$, then $Y = \mu + \sigma X$ is a $KwSN$ random variable with vector of parameters $\boldsymbol{\xi} = (\mu, \sigma, \lambda, a, b)$ or $Y \sim KwSN(\mu, \sigma, \lambda, a, b)$. Hereafter, we will denote by SN the skew-normal, by $N(0, 1)$ the normal, by KwN the Kumaraswamy-normal and by Kw the Kumaraswamy distributions.

We now give some simple properties of $KwSN(\lambda, a, b)$ density in (3).

Property 1. *Let $X \sim KwSN(\lambda, a, b)$.*

(a) *If $a = b = 1$, then $X \sim SN(\lambda)$.*

(b) *If $\lambda = 0$, then $X \sim KwN(a, b)$.*

(c) *If $a = b = 1$ and $\lambda = 0$, then $X \sim N(0, 1)$.*

(d) *If $a = \frac{1}{2}$, $b = 1$ and $\lambda = 1$, then $X \sim N(0, 1)$.*

(e) *If $a = 1$, $b = \frac{1}{2}$ and $\lambda = -1$, then $X \sim N(0, 1)$.*

(f) *Let $Y = \Phi(X; \lambda)$, then $Y \sim Kw(a, b)$.*

(g) *Let $Y = \Phi(X; \lambda)^a$, then $Y \sim Kw(1, b)$. Let $Z = 1 - Y$, then $Z \sim Kw(b, 1)$.*

(h) *As $\lambda \to +\infty$, the $KwSN$ density tends to the Kumaraswamy half-normal density (Cordeiro et al., 2012).*

(i) *If $b = 1$, then $X \sim BSN(\lambda, a, 1)$.*

Proof. The results follow immediately by taking into account expression (3) and from elementary properties of the skew-normal distribution.

Graphical displays of the $KwSN$ density with various combination of the parameters are shown in Figure 1. It should be noted that the $KwSN$ density does not exhibit bimodality for any parameter values, unlike the BSN one which is unimodal or bimodal according to the region in which (λ, a, b) lie (Mameli and Musio, 2013).

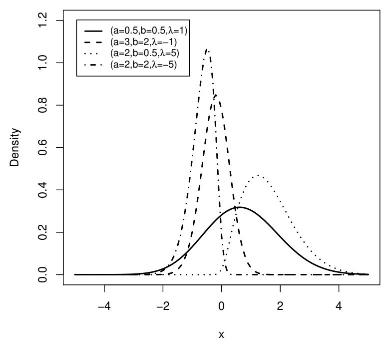

Figure 1: The $KwSN(\lambda, a, b)$ for different values of λ, a and b.

As a result of the skewing mechanism introduced by Ferreira and Steel (2006), the $KwSN(\lambda, a, b)$ density can be represented as

$$g^K_{\Phi(x;\lambda)}(x; \lambda, a, b) = \phi(y)p(\Phi(y)), \quad -\infty < y < \infty$$

with $p(\cdot)$ on $(0, 1)$ given by

$$p(u; \lambda, a, b) = 2ab\Phi(\lambda\Phi^{-1}(u)) \left(\Phi(\Phi^{-1}(u); \lambda)\right)^{a-1} \left(1 - \Phi(\Phi^{-1}(u); \lambda)^a\right)^{b-1}. \quad (4)$$

In view of this, the $KwSN(\lambda, a, b)$ density is a member of the unified skewed distributions of Abtahi et al. (2011). Therefore a stochastic representation of $X \sim KwSN(\lambda, a, b)$ follows from Proposition 1 in Abtahi et al. (2011). In particular, the following property is obtained.

Property 2. *Let U be a standard normal variable and let V be independent of U with pdf p on $(0, 1)$ given by equation (4).*

- *When $W = V - \Phi(U)$, the conditional distribution of U given $(W = 0)$ is $KwSN(\lambda, a, b)$.*

- *Let $X \sim KwSN(\lambda, a, b)$. Then $\Phi(X) \stackrel{d}{=} V$.*

Given the closed form of the cumulative distribution function, the $KwSN$ distribution can be easily simulated by using the probability integral transformation, or by using Property 1 item (f). Supposed that $a, b \geq 1$, another method for generating samples from the $KwSN$ is given by the algorithm of acceptance-rejection in Nadarajah et al. (2012).

2.2. Moment generating function and moments

In this section we provide the moment generating function of the $KwSN(\lambda, a, b)$ distribution. The proofs of the following two propositions are given in the supplementary material.

Proposition 1. *The moment generating function of $X \sim KwSN(\lambda, a, b)$ is given by*

$$M_X(t) = 2abe^{\frac{t^2}{2}} E_Z \left[(\Phi(Z;\lambda))^{a-1} (1 - \Phi(Z;\lambda)^a)^{b-1} \Phi(\lambda Z) \right],$$

where $Z \sim N(t, 1)$.

We also get a recursive formula for the $k-th$ moment.

Proposition 2. *Let $k \in \mathbb{N}$ and $k \geq 1$. If $X \sim KwSN(\lambda, a, b)$, with $b > 1$ then*

$$\begin{aligned} E_X\left[X^k\right] &= (k-1)E_X\left[X^{k-2}\right] + \lambda E_X\left[X^{k-1}\frac{\phi(\lambda X)}{\Phi(\lambda X)}\right] + \\ &+ (a-1)E_X\left[X^{k-1}\frac{\phi(X;\lambda)}{\Phi(X;\lambda)}\right] - baE_V\left[V^{k-1}\frac{\phi(V;\lambda)}{\Phi(V;\lambda)}\right], \end{aligned} \tag{5}$$

where $V \sim KwSN(\lambda, a, b-1)$ is independent from X.

Moments of the $KwSN$ distribution, as the ones of BSN (Mameli and Musio, 2013), have not a closed form. However, they can be calculated numerically by using any numerical computing environment. The classical skewness (γ_1) and excess kurtosis (γ_2), and the Bowley's skewness (B) and Moors' kurtosis (M) of a $KwSN(\lambda, a, b)$ random variable are numerically computed for various values of the parameters; see table 1 of the supplementary material. See Alexander et al. (2012) for the treatment of the Bowley's skewness and Moors' kurtosis in the

Generalized Beta-Generated family of distributions of which the Kumaraswamy skew-normal distribution is a member. It is worth to remark that the index of skewness γ_1 is restricted to the interval $(-0.995, 0.995)$ of the SN distribution, while the index of kurtosis γ_2 lies in the range $(0, 0.869)$ (Azzalini (1985)). Then from table 1 of the supplementary material it can be inferred that the Kumaraswamy skew-normal distribution exhibits more flexibility than the SN one.

Graphical displays with various combinations of parameters, given in the supplementary material, show the numerical behaviour of the mean and of the skewness γ_1 as functions of b for various values of a and λ.

The following theorem presents expressions for moments of the $KwSN$ when the parameters a and b lie in the set of positive integers.

Theorem 1. *Let $X \sim KwSN(\mu, \sigma, \lambda, a, b)$ for integers values of a and b, then*

$$\begin{aligned} E\left[X^n\right] &= \mu^n + 2ab\mu^n \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \sum_{i=1}^{n} \binom{n}{i} \left(\frac{\sigma}{\mu}\right)^i \cdot \\ &\cdot \left\{ \sum_{k=0}^{a(j+1)-1} (-1)^k \binom{a(j+1)-1}{k} I_{i,k,\lambda} + (-1)^i I_{i,a(j+1)-1,-\lambda} \right\}, \end{aligned}$$

where

$$I_{i,k,\lambda} = \int_0^\infty z^i \phi(z) \Phi(\lambda z)(1 - \Phi(z;\lambda))^k dz.$$

Proof. The proof parallels that of theorem 1 in Gupta and Nadarajah (2005) and is given in the supplementary material.

2.3. The $KwSN(1, n, b)$

As previously mentioned, moment generating function and moments of $KwSN(\lambda, a, b)$ are in general difficult to evaluate analytically. However, we will show that moments and moment generating function of the sub-model $KwSN(1, n, b)$, obtained by considering $\lambda = 1$ and an integer parameter $a = n$, can be easily derived by using results on the Balakrishnan skew-normal distribution (Sharafi and Behboodian, 2008). Throughout the section, we shall denote by $X \sim SNB(l, \alpha)$ a Balakrishnan skew-normal random variable with parameters l and α and by

$c_l(\alpha)$ the associated normalizing constant. The proofs of the following two theorems are given in the supplementary material.

Theorem 2. *The moment generating function of* $X \sim KwSN(1, n, b)$ *is*

$$M_X(t) = 2nb \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} e^{\frac{t^2}{2}} E[\Phi(V)^{2n(1+j)-1}],$$

where $V \sim N(t, 1)$.

The moments are given by the following recursive formula.

Theorem 3. *Let* $X \sim KwSN(1, n, b)$*, then*

$$E\left[X^k\right] = \frac{b}{\sqrt{\pi} 2^{\frac{k+1}{2}}} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{b-1}{j}}{(1+j)} \left\{ \frac{2n(1+j)[2n(1+j)-1]}{c_{(2n(1+j)-2)}\left(\frac{1}{\sqrt{2}}\right)} E\left[W^{k-1}\right] + (k-1) E\left[Y^{k-2}\right] \right\},$$

where $W \sim SNB\left((2n(1+j)-2), \frac{1}{\sqrt{2}}\right)$ *and* $Y \sim SNB\left((2(n+j)-1), 1\right)$.

Similar properties hold also for the moments of the $BSN(1, n, b)$, sub-model of the $BSN(\lambda, a, b)$; see Mameli (2012) and Mameli and Musio (2015). Moreover, note also that moments of the SNB requires the evaluations of the normalizing constant $c_l(\alpha)$, whose approximated values can be found in Steck (1962).

2.4. Distributional properties

Here, we focus on distributional properties of the $KwSN$ distribution and on the $Kw - F$ family.

Let us first start with the following property whose proof is given in the supplementary material:

Property 3. *Let* $X \sim KwSN(\lambda, a, b)$ *be independent of* $Y \sim KwSN(\lambda, a, d)$ *then* $X| (Y \geq X) \sim KwSN(\lambda, a, b + d)$*, where* $a, b, d > 0$.

The proofs of the following properties are quite similar to that of Property (3) and are therefore omitted.

In a similar fashion, we get the following property:

Property 4. *Let* $X \sim KwSN(\lambda, a, 1)$ *be independent of* $Y \sim KwSN(\lambda, c, 1)$. *Then* $X|(Y \leq X) \sim KwSN(\lambda, a + c, 1)$*, where* $a, c > 0$.

Note that the property follows easily by Property 1 item (i) and Mameli and Musio (2013). In the special case that $Y = Y_{(n)}$, the largest order statistic, the property permits to generate a $KwSN(\lambda, n, 1)$ by using the acceptance-rejection technique; see Mameli and Musio (2013) and references therein.

The results presented in Properties 5 and 6 are generalizations to the class of the Kumaraswamy generalized family $Kw - F$ (Cordeiro and de Castro, 2011), defined at equation (2), of Properties 3 and 4, respectively. The parameters a, b, c and d are positive real parameters which control skewness and tail weights of the family.

Property 5. *Let* $X \sim Kw - F(a, b)$ *be independent of* $Y \sim Kw - F(a, d)$ *then* $X|\,(Y \geq X) \sim Kw - F(a, b + d)$*, where* $a, b, d > 0$*.*

Property 6. *Let* $X \sim Kw-F(a, 1)$ *be independent of* $Y \sim Kw-F(c, 1)$*. Then* $X|(Y \leq X) \sim Kw - F(a + c, 1)$*, where* $a, c > 0$*.*

2.5. An interesting proposition

The Kumaraswamy skew-normal distribution, as well as the BSN distribution, leads to the normal distribution for three different parameter sets; see properties outlined in section 2.1. It can be inferred from these properties that identifiability problems can occur under the null hypothesis of normality, therefore parameters cannot be uniquely determined. Then it would be desirable to determine all values of the parameters a, b and λ which yield the normal distribution. Due to the tractability of the $KwSN$ density, next result shows that it is possible to determine all these possible sets of parameters. The proof of this proposition is given in the supplementary material.

Proposition 3. *Let* $X \sim KwSN(\lambda, a, b)$ *and* $\boldsymbol{\xi} = (\lambda, a, b)$*. The distribution of* X *reduces to a normal distribution if and only if one of the following conditions holds:*

1. $\boldsymbol{\xi} = (0, 1, 1)$*;*
2. $\boldsymbol{\xi} = (1, \frac{1}{2}, 1)$*;*
3. $\boldsymbol{\xi} = (-1, 1, \frac{1}{2})$.

3. Estimation

As we have seen in the previous section, the parameters representing the true null distribution are not unique in the Gaussian case and classical likelihood result does not applies (see e.g. Liu and Shao (2003) and references therein). Hence, we just consider a special subclass of this family by choosing $b = \frac{1}{a}$, this choice leads to a model which under the null hypothesis of normality is described only by the parameters $a = 1, b = 1$ and $\lambda = 0$.

Consider a sample $x_1, \ldots, x_N$ from the $KwSN(a, \frac{1}{a}, \mu, \sigma, \lambda)$ density. The log-likelihood function $l(\boldsymbol{\xi})$ for the vector of parameters $\boldsymbol{\xi} = (a, \mu, \sigma, \lambda)$ is

$$l(\boldsymbol{\xi}) = -N \log \sigma + \sum_{i=1}^{N} \left[\log\left(\phi\left(z_i; \lambda\right)\right) + (a-1)\log\left(v_i\right) + \left(\frac{1}{a} - 1\right) \log\left(1 - v_i^a\right) \right],$$

where $z_i = \frac{x_i - \mu}{\sigma}$ and $v_i = \Phi\left(z_i; \lambda\right)$.

The components of the score vector are given in the supplementary material.

Estimation of the parameters could be carried out, for instance, through the numerical procedure *nlminb* available in the computing R environment.

3.1. Illustrative example

We analyse the data set `ais` available in the `R`-package `sn` (see Azzalini (2014), Cook and Weisberg (1994)), which consists of different measurements on 202 Australian athletes. Here, we concentrate on the variable measuring the body mass index (BMI). We note that the data have positive skewness (0.9465) and positive excess kurtosis (2.1835). We fit the BSN, the $KwSN$, and the SN distributions to the BMI data and we use the Akaike Information Criterion (AIC) to compare the three models. Table 1 displays parameter estimates with standard errors in parenthesis for the three distributions as well as the values of the AIC and of the log-likelihood ($log - lik$). In line with the AIC criterion, the $KwSN$ distribution furnishes the best model. Figure 2 illustrates the histogram of the data in conjunction with the fitted densities and the kernel density estimate. Even though, the $KwSN$ and the BSN show very similar fit, the estimates of the unknown parameters of BSN distribution display larger variability than the corresponding estimates of the $KwSN$. To investigate the variability of the estimates we conducted a simulation study, whose results are reported in the supplementary material (Tables 2–4). Estimation of parameters was based on 2,000 Monte Carlo replications from three simulated samples of size 500 from the skew-normal distribution for different values of the parameters. The simulation study shows the performances of the estimates of the parameters along with the standard errors in parentheses for the $KwSN$, for the BSN and for the SN distributions. The study has highlighted that the estimates of the parameter λ of the BSN distribution display slightly more variability than the corresponding estimates of the $KwSN$.

Moreover, a further simulation study, not reported, shows that when the parameter a is far from 1.5 the maximization algorithm does not attain the global maximum and converges to a local maximum both for the $KwSN$ and the BSN distributions; the convergence of the algorithm to the global maximum depends mainly on the choice of the starting values.

Table 1: Parameter estimates for the BMI data set with standard errors in parenthesis and the associated AIC and log-likelihood ($log - lik$) values for each of the models considered

	a	μ	σ	λ	AIC	$log - lik$
BSN	0.1622 (0.1139)	24.1582 (1.0280)	21.1714 (19.1637)	20.3588 (23.6592)	980.3090	−486.1545
$KwSN$	0.2996 (0.0892)	24.5336 (1.2452)	15.2074 (11.5439)	10.4375 (8.5670)	980.2804	−486.1402
SN	–	19.9697 (0.3288)	4.1327 (0.3142)	2.3126 (0.5135)	986.1988	−490.0994

4. Final remarks

In this paper, we study some structural properties of a new generalization of the skew-normal distribution, called the Kumaraswamy skew-normal distribution, which represents a valid alternative to the Beta skew-normal one (Mameli and Musio, 2013). Both distributions have the normal and the skew-normal distributions as special cases. The Kumaraswamy skew-normal model, as well as the Beta skew-normal one, presents problems of identifiability under the null hypothesis of normality. Due to the tractability of the $KwSN$, it is possible

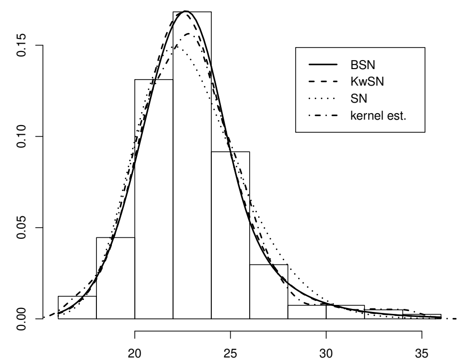

Figure 2: Histogram of the BMI data set for 202 Australian athletes with superimposed the estimated densities: BSN (solid line), $KwSN$ (dashed line), SN (dotted line), and kernel estimate (dot-dashed).

to determine all the sets of parameters for which the density reduces to the normal. There is not an analogous result for the Beta skew-normal density. Investigation of the maximum likelihood estimator under loss of identifiability requires deeper work and will be presented in a future paper. Furthermore, it could be of interest to study the behaviour of the maximum likelihood estimator in relation with the inferential problems of the skew-normal distribution.

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