RENDICONTI

PER GLI

STUDI ECONOMICI

QUANTITATIVI

*Numero speciale in onore
di Giovanni Castellani*

Settembre 2005

Non–linear modelling of bivariate comovements in asset prices

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Abstract. The phenomenon of comovements among asset prices has received a lot of attention for several reasons (see, for some examples, section 1). The increasing interest in this topic has been the reason of the production of a large number of contributions. In this paper we propose an investigating methodology for the non–linear modelling of bivariate comovements. Our approach leaves the ones presented in the recent literature. In fact, our approach, which is articulated in three steps, allows the evaluation and the statistical testing of non–linearly driven comovements between two given random variables. Moreover, when such an (unknown) bivariate dependence relationship is detected, our approach allows also to provide a polynomial approximation of it. Finally, we apply our three–steps methodology to some energy asset prices time series traded in the U.S.A.. The goodness of the results is encouraging given the novelty of the proposed investigating approach.

Keywords. Comovement, asset price, bivariate dependence, non–linearity, comonotonicity, t-test, polynomial approximation, energy asset.

1 Introduction

The issue regarding the phenomenon of comovements among asset prices has received a lot of attention for several reasons:

- firstly, the knowledge of dependence relationships among the prices of given stocks allows to obtain information about a not–ready–to–observe stock price by suitably using the ready–to–observe ones. Moreover, it make also possible cross–hedging and cross-speculation approaches;
- secondly, the presence, or less, of dependence in form of correlation among the prices of assets traded in different countries is of interest to investors who wish to allocate their capitals in mean–variance portfolios since, as known, international diversification strategies work well when the considered markets are little integrated;

- thirdly, dependence among stock prices traded in different countries is of interest to policy makers as such comovements can affect domestic consumptions;
- last, scholars and various institutions are interested in establishing and in investigating the extent of integration level among financial markets.

The increasing interest in the topic of comovements in asset prices has been the reason of the production of a large number of contributions in the specialized literature. In the next section we provide a short survey of the more recent of such contributions. In particular, in most of these studies the various authors make use of investigating approaches mainly based on autoregressive heteroskedastic (ARCH) models, error correction models (ECMs), generalized ARCH (GARCH) models, Granger causality based tests, multivariate cointegrations, structural vector autoregression (VAR) systems, lag–augmented VAR (LA–VAR) systems, forecast error variance decomposition (VDC) approaches, and vector error–correction models (VECMs).

As far as concerns the investigating methodology we propose in this paper, it leaves the approaches listed above. In fact, our approach, which is articulated in three steps, allow the evaluation and the statistical testing of non–linearly driven comovements between two given random variables. Moreover, when such an (unknown) bivariate dependence relationship is detected, our approach allows also to provide a polynomial approximation of it.

The remainder of this paper is organized in the way which follows. As premised, in the next section we present a short review of the recent literature. In section **3** we propose in detail our three–steps methodology. In section **4** we provide the results of some applications of the proposed methodology to time series of the prices of energy assets traded in U.S.A.. Finally, In section **5** we give some concluding remarks.

2   A short review of the recent literature

In this section we give a short survey of the recent literature about the comovements among assets prices.

Before to begin, notice that a significant percentage of the published contributions concern cross-country dependence relationships.

In [6] the mechanism of international transmission of stock prices movements is investigated by using a nine–market VAR system. In particular, the authors trace out the dynamics of the responses in a given market to the innovations in another given one. In [5], by using univariate and multivariate GARCH models, it is shown that the prices of several (to all apparencies) unrelated markets reveal a persistent tendency to comove, even after accounting for the effects of macroeconomic shocks. In [11] long–term and short–term dependence relationships among the prices of six agricultural futures traded at the Chicago Board of Trade are analyzed by using the ECM. In [7] the investigation of interdependencies among stock prices is performed by using a LA–VAR system based

approach. A significant advantage of this methodology consists in the fact that it can be applied regardless of the presence, or less, of cointegration among the considered stock prices. In [1] cointegration among stock prices traded in different countries is investigated. In particular, the authors put in evidence that the likelihood ratio tests of Johansen are sensitive to the specification of the time lag amplitude in the VAR system.

Some other methodologies which are worth while mentioning are the ones able to detect the presence, or less, of common cycles among asset prices. In [3] a cointegration technique is utilized for testing the presence of long–run common trends among stock prices and the interest rate, and co-dependence analyses are performed for investigating the presence and the features of short–run common cycles among the same quantities. In [4] linear and non–linear Granger causality based tests are used to examine the dynamical dependence relationships between spot and future prices. Finally, in [13] proper measures of dependence among European stock markets are evaluated by using the multivariate extreme value theory.

3 Our three–steps methodology

In this section we present in detail our methodology for the non–linear evaluation of bivariate comovements. Since our approach is (softly) based on the concept of comonotonicity, before of all we spend some words about this notion.

Comonotonicity is one of the strongest measure of dependence existing among random variables. Limiting our interest to the bivariate case, given two random variables $X_1(t)$ and $X_2(t)$, both defined in $[t_0, t_1]$ with $t_0 < t_1$, they are said to be comonotonic if and only if:[1]

$$[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] \geq 0 \ \forall \ t_2, t_3 \ : \ t_2 \neq t_3 \ \wedge \ t_2, t_3 \in [t_0, t_1].$$

A few remarks about this relationship:

- two random variables are comonotonic if and only if they always vary over the support (time) in the same direction, besides the quantitative laws describing the dynamic behaviour of each of them;
- comonotonicity is an ON/OFF concept, in fact it is sufficient the existence of a unique pair t_2 and t_3 for which $[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] < 0$ to state that X_1 and X_2 are not comonotonic. Of course, in such a case it should be hard to uphold that, as $X_1(t)$ and $X_2(t)$ are not more comonotonic, they are also not more dependent in some sense (in profiling our approach we start from this latest remark).

Our methodology is articulated in three steps. Before to present in detail each of them, we give a brief description of their contents:

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[1] For other equivalent definitions of comonotonicity see [8], [9] and [15].

- in the first step we propose a simple index able to evaluate any intermediate degree of bivariate dependence from full countermonotonicity[2] to full comonotonicity, and we provide some theoretical results about it;

- as this simple index provides only a point estimation of the considered bivariate dependence, in the second step we propose a procedure by which to test the statistical meaningfulness of the index itself;

- once the statistical meaningfulness of the simple index has been proved, in the third step we propose an algorithm able to provide a polynomial approximation of the unknown bivariate dependence relationship.

3.1 The simple index

Let we start by considering two discrete–time time series, $\{X_1(t),\ t = t_1,\ \ldots, t_N\}$ and $\{X_2(t), t = t_1, \ldots, t_N\}$. The simple index we propose for evaluating the bivariate dependence between the random variables $X_1(t)$ and $X_2(t)$ is defined as follows:

$$\delta_{1,2} = \frac{1}{N-1} \sum_{t=t_2}^{t_N} \Delta(t)_{1,2},$$

$$\Delta(t)_{1,2} = \begin{cases} -1 & \text{if } [X_1(t) - X_1(t-1)][X_2(t) - X_2(t-1)] < 0 \\ 1 & \text{if } [X_1(t) - X_1(t-1)][X_2(t) - X_2(t-1)] \geq 0 \end{cases}. \tag{1}$$

Some remarks about this index:

- it is trivial to prove that $\delta_{1,2} \in [-1, 1]$. In particular, the two random variables are countermonotonic if and only if $\delta_{1,2} = -1$, and are comonotonic if and only if $\delta_{1,2} = 1$;

- beyond the property reported in the previous point (property of normalization of the first type), it is also trivial to prove that $\delta_{1,2}$ is defined for every pair of discrete–time time series (property of existence), and that $\delta_{1,2} = \delta_{2,1}$ (property of symmetry). Therefore, $\delta_{1,2}$ is a scalar measure of dependence in the sense illustrated in [14] at section 6;

- the fact that $\delta_{1,2}$ belongs to $[-1, 1]$ makes this index of dependence directly comparable with the well known and widely used Bravais–Pearson linear correlation coefficient $\rho_{1,2}$.[3]

As far as theoretical properties between $\delta_{1,2}$ and $\rho_{1,2}$ are concerned, we give the proposition which follows.

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[2] Two random variables $X_1(t)$ and $X_2(t)$, both defined in $[t_0, t_1]$ with $t_0 < t_1$, are said to be countermonotonic if and only if $[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] < 0$ for all t_2, t_3 such that $t_2 \neq t_3$ and $t_2, t_3 \in [t_0, t_1]$.

[3] Also $\rho_{1,2}$ is a scalar measure of dependence in the sense illustrated in [14] at section 6.

Proposition 1. *Let $f(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}_+$ be the bivariate dependence relationship between $X_1(t)$ and $X_2(t)$, i.e. $X_1(t) = f(X_2(t)) + \varepsilon(t)$, where $\varepsilon(t)$ has the usual meaning, and let $f(\cdot)$ be infinite times derivable in $m_2 = \mathbb{E}(X_2(t))$. If*

$$\frac{f^{(i)}(m_2)}{i!} \binom{i}{i-j} (-m_2)^{i-j} = 0 \ \forall \ j = 2, \ldots, +\infty, \tag{2}$$

where $f^{(i)}(\cdot)$ indicates the i-th derivatives of $f(\cdot)$, then the bivariate dependence relationship is affine.

Proof. As $f(\cdot) \in C^\infty$, we can expand it in Taylor's series about m_2 as follows:

$$\begin{aligned}
f(X_2(t)) &= \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} (X_2(t) - m_2)^i \\
&= \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \sum_{j=0}^{i} \binom{i}{j} X_2^{i-j}(t)(-m_2)^j.
\end{aligned} \tag{3}$$

After some algebraic manipulations, we can rewrite equation (3) as follows:

$$f(X_2(t)) = \sum_{j=0}^{+\infty} \left[\sum_{i=j}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i-j} (-m_2)^{i-j} \right] X_2^j(t). \tag{4}$$

Now, by substituting relationships (2) into (4) we obtain the following affine bivariate dependence relationship between $X_1(t)$ and $X_2(t)$:

$$\begin{aligned}
X_1(t) = &\sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i} (-m_2)^i + \\
&+ \left[\sum_{i=1}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i-1} (-m_2)^{i-1} \right] X_2(t) + \varepsilon(t). \ \square
\end{aligned} \tag{5}$$

Notice that, if relationship (2) were extended also to $j = 1$, then relationship (5) should begin

$$X_1(t) = \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i} (-m_2)^i + \varepsilon(t),$$

i.e. there not should be more any dependence relationship between $X_1(t)$ and $X_2(t)$, *i.e.* $X_1(t)$ and $X_2(t)$ should be independent.

As premised, the simple index we proposed here provides only a point estimation of the investigated bivariate dependence. In order to overcome this drawback, in the next subsection we propose a procedure able to statistically test the meaningfulness of the index itself.

3.2 The testing procedure

The "philosophy" of the procedural approach we propose here for testing the statistical meaningfulness of $\delta_{1,2}$ is similar to the one of the procedural approach proposed in [10].

In the remainder of this subsection we present in detail our testing procedure in the itemized form which follows:

- firstly, we define the random variable $\delta_{S;1,2}$ as the index (1) applied to the time series $\{X_1(t), t = t_1, \ldots, t_N\}$ and $\{X_2(t), t = t_1, \ldots, t_N\}$ once both the time series have been shuffled according to the same independent and identical uniform distribution (notice that, as the shuffling should destroy any dependence relationship between $X_1(t)$ and $X_2(t)$, $\delta_{S;1,2}$ should be equal to 0);

- secondly, we define the quantity $\Delta\delta(1) = \delta_{1,2} - \delta_{S;1,2}$, and generate the series $\{\Delta\delta(j), j = 1, \ldots, M\}$ by shuffling $\{X_1(t), t = t_1, \ldots, t_N\}$ and $\{X_2(t), t = t_1, \ldots, t_N\}$ as described in the previous point for M times (notice that, if $X_1(t)$ and $X_2(t)$ were $\delta_{1,2}$−dependent, then $\Delta\delta(1)$ should be different from 0);

- thirdly, we determine the estimations of the sample mean and of the sample standard deviation of $\Delta\delta$, $m_{\Delta\delta}$ and $s_{\Delta\delta}$ respectively, as follows:

$$m_{\Delta\delta} = \frac{1}{M}\sum_{j=1}^{M}(\delta_{1,2} - \delta_{S;1,2}(j)) \quad \text{and} \quad s_{\Delta\delta} = \sqrt{\frac{1}{M}\sum_{j=1}^{M}(\delta_{S;1,2}(j) - m_{\Delta\delta})^2};$$

- fourthly, recalling from basic statistics that

$$\frac{m_{\Delta\delta} - (\delta_{1,2} - \delta_{S;1,2})}{s_{\Delta\delta}\Big/\sqrt{M}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad M \to +\infty,$$

for M large enough, we can perform the following bilateral t−test:

$$\begin{cases} H_0\colon m_{\Delta\delta} = 0, & \text{i.e. } X_1(t) \text{ and } X_2(t) \text{ are independent} \\ H_1\colon m_{\Delta\delta} \neq 0, & \text{i.e. } X_1(t) \text{ and } X_2(t) \text{ are } \delta_{1,2}\text{−dependent} \end{cases} \tag{6}$$

in particular, the acceptance interval for the null hypothesis is $((-s_{\Delta\delta} \cdot t_{\alpha/2})\big/\sqrt{M-1}, (s_{\Delta\delta} \cdot t_{\alpha/2})\big/\sqrt{M-1})$, where $t_{\alpha/2}$ is the value taken by a t−distributed random variable in correspondence of a pre−established confidence interval α for given degrees of freedom;

- finally, if the null hypothesis presented in the previous point is rejected, then we perform two more unilateral t−tests in order to verify whether the $\delta_{1,2}$−dependence between $X_1(t)$ and $X_2(t)$ is negative or positive. In particular, both such tests differ from the one introduced in the previous point only in the alternative hypothesis, which is $H_1\colon m_{\Delta\delta} < 0$ in the negative $\delta_{1,2}$−dependence case, and is $H_1\colon m_{\Delta\delta} > 0$ in the positive $\delta_{1,2}$−dependence case.

Notice that, in order to reduce the amplitude of the acceptance intervals, i.e. to reduce $s_{\Delta\delta}\big/\sqrt{M}$ to $s_{\Delta\delta}\big/\big(c\sqrt{M}\big)$, with $c > 1$, one has to increase M to $\lceil c^2 M \rceil$.[4] Because of that, profitable applications of our methodology sometimes could be time−consuming.

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[4] $\lceil \cdot \rceil$ is the minimal integer which exceeds the value taken by the expression inside the notation itself.

3.3 The polynomial approximation

If at the end of the testing procedure the null hypothesis has been rejected in favour of the negative/positive $\delta_{1,2}$−dependence between $X_1(t)$ and $X_2(t)$, then we begin to model in analytical way the unknown bivariate dependence relationship $X_1(t) = f(X_2(t)) + \varepsilon(t)$. In particular, we search for a polynomial approximation of $f(\cdot)$ which is a properly truncated version of the equation (4), i.e.

$$f(X_2(t)) = \sum_{j=0}^{J} a_j X_2^j(t) + r(J+1), \tag{7}$$

where J is the truncation order of the Taylor's series (4), $a_j = \sum_{i=j}^{J} \frac{f^{(i)}(m_2)}{i!} \cdot \binom{i}{i-j}(-m_2)^{i-j}$, and $r(J+1)$ is a suitable remainder function.

Of course, in such an approach a crucial role is played by J. In order to detect its "optimal" value, we propose the following algorithm whose search procedure is based on a standard cross–validation technique, as suggested for empirical work in [12] at section 4:

- we begin by considering as starting data set D the discrete–time bivariate time series $\{(X_1(t), X_2(t)), t = t_1, \ldots, t_N\}$;
- secondly, we suitably split D into two data subsets, the learning one D_L and the validation one D_V, such that $D_L \cup D_V = D$ and $D_L \cap D_V = \emptyset$;[5]
- thirdly, we consider a finite series of polynomials of kind (7) with $J = 0, \ldots,$ $\overline{J}$, where $\overline{J}$ is a pre–established integer value;
- fourthly, for each of the polynomials considered in the previous point we estimate the parameters $a_0, \ldots, a_J$ via ordinary least square regression by using the data subset D_L, and evaluate the index $\delta_{1,2}$ between $\widehat{X}_1(t) = \sum_{j=0}^{J} \widehat{a}_j X_2^j(t)$ and $X_2(t)$ by using the data subset D_V;[6]
- finally, we choose as "best" approximating polynomial the one to which is associated the highest absolute value of $\delta_{1,2}$.

Notice that the fact of identifying the "optimal" approximating polynomial by using a cross–validation approach, i.e. to perform the ordinary least square by using the learning data subset and to evaluate the validation criterion $|\delta_{1,2}|$ by using the validation data subset, allows to strongly avoid overspecialization of the polynomial itself.

4 Applications to energy asset prices time series

In this section we give the results of some applications of our three–steps methodology to the prices time series of some energy asset traded in the U.S.A..

In general terms, for each application we act as follows:

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[5] The way in which to suitably split D is made clear in subsection **4.2**.

[6] $\widehat{\cdot}$ indicates the estimator of the quantity below.

- we start by considering the discrete–time bivariate time series $\{(X_1(t),\ X_2(t)),\ t = t_1,\ \ldots,\ t_N\}$;

- from the time series introduced in the previous point we split the chronologically last 10 per cent of its realizations in order to utilize them as forecasting data subset D_F at the end of the application for performing a rough out–of–sample check. We use the remaining 90 percent of the discrete time bivariate time series as the starting data set D;

- we split D into the learning data subset D_L (the chronologically first 70 per cent of its realizations) and the validation data subset D_V (the chronologically last 30 per cent of its realizations);[7]

- finally, we perform our methodology by using D_L and D_V.

4.1 The data

Each discrete–time univariate time series we utilize here is constituted by $2,026$ daily spot closing prices of three energy assets traded in U.S.A.: the crude oil, the gasoline, and the heating oil. Such prices have been collected from January 3, 1884 to February 6, 2002. In the remainder of this subsection and in the next one, we refer to these time series respectively as $\{X_{CO}(t),\ t = t_1,\ \ldots,\ t_{2,026}\}$, $\{X_G(t),\ t = t_1,\ \ldots,\ t_{2,026}\}$, and $\{X_{HO}(t),\ t = t_1,\ \ldots,\ t_{2,026}\}$.

As far as concerns the discrete–time bivariate time series whose non–linear comovements we investigate here, we consider all the ones forecoming from the simple disposition of the discrete–time univariate time series listed in the previous paragraph, i.e. $\{(X_{CO}(t),\ X_G(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$, $\{(X_{CO}(t),\ X_{HO}(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$, $\{(X_G(t),\ X_{CO}(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$, $\{(X_G(t),\ X_{HO}(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$, $\{(X_{HO}(t),\ X_{CO}(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$, and $\{(X_{HO}(t),\ X_G(t)),\ t = t_1,\ \ldots,\ t_{2,026}\}$.

Notice that, given the percentages set in the previous itemization with regard to the data subsets implied in each application, the cardinalities of these same data subset are the following: $\#\{D_L\} = 1,277$, $\#\{D_V\} = 547$, and $\#\{D_F\} = 202$.

4.2 The results

As premised, here we provide and illustrate the results of the applications of our three–steps methodology to the discrete–time bivariate time series listed in the previous subsection. The exposition of these results is organized in two tables, and in some figures.

As far **Table 1** is concerned, before to report it we need to specify the content of each of its columns:

- the first column indicates the two random variables specifying the discrete–time bivariate time series which has investigated;

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[7] The percentages we set for D_L and D_V are the ones usually utilized in several empirical works using cross–validation techniques (see, for example, [2] and the references therein).

- the second column reports the value of the simple index $\delta_{i,j}$, with $i, j \in \{CO, G, HO\}$ and $i \neq j$, evaluated on the learning data subset D_L (see, for more details, subsection **3.1**);
- the third column provides the response of the bilateral t−test (6):[8] label "A" or label "R" for, respectively, the acceptance or the rejection of the null hypothesis (see, for more details, subsection **3.2**);
- if the null hypothesis of the bilateral t−test (6) is rejected, then the fourth column gives the response of the check, based on two more unilateral t−tests,[9] whether the $\delta_{1,2}$−dependence between the two investigated univariate time series is negative, label "N", or positive, label "P" (see, for more details, again subsection **3.2**);
- the fifth column gives the value of the Bravais–Pearson linear correlation coefficient $\rho_{i,j}$, with $i, j \in \{CO, G, HO\}$ and $i \neq j$, evaluated on the learning data subset D_L (we report the value of this coefficient for possible comparisons).

Finally, we recall that the property of symmetry holds for the simple index (1), i.e. $\delta_{i,j} = \delta_{j,i}$ for all i, j such that $i, j \in \{CO, G, HO\}$.

Table 1.

Random variables	$\delta_{i,j}$	Bilateral t−test	Check on the $\delta_{1,2}$−dep.	$\rho_{i,j}$
$X_{CO}(t)$, $X_G(t)$	0.35407	R	P	0.94139
$X_{CO}(t)$, $X_{HO}(t)$	0.39259	R	P	0.91178
$X_G(t)$, X_{HO}	0.48642	R	P	0.85118

A few remarks about the results reported in **Table 1**:

- the fact that $\delta_{i,j}$ is statistically significantly different from 0 for all i, j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the third column of **Table 1**) indicates the existence of a bivariate dependence relationship between $X_i(t)$ and $X_j(t)$ for all i, j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$;
- recalling that the Bravais–Pearson coefficient measures only the linear correlation, the fact that $\delta_{i,j}$ is significantly different from $\rho_{i,j}$ for all i, j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the fifth column of **Table 1**) puts in evidence the presence of non–linearities in the bivariate dependence relationships presented in the previous point;
- the fact that $\delta_{i,j}$ and $\rho_{i,j}$ are both positive for all i, j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the fifth column of **Table 1** again) can be interpreted as an indicator of the positiveness of the dependence

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[8] In performing this bilateral test we set $M = 100$ and $\alpha = 5\%$.

[9] Also in performing these unilateral tests we set $M = 100$ and $\alpha = 5\%$.

between $X_i(t)$ and $X_j(t)$ for all i, j such that i, $j \in \{CO, G, HO\}$ and $i \neq j$.

Also as far **Table 2** is concerned, before to report it we need to specify the content of each of its columns:

- the first column indicates the two random variables specifying the discrete-time bivariate time series which has investigated;
- the second column provides the estimation of "best" polynomial approximation of the unknown bivariate dependence relationship between $X_i(t)$ and $X_j(t)$ for all i, j such that i, $j \in \{CO, G, HO\}$ and $i \neq j$ (see, for more details, subsection **3.3**).

Table 2.

Random variables	Polynomial approximation
$X_{CO}(t)$, $X_G(t)$	$X_{CO}(t) = 1.68731 + 31.36198 X_G(t)$
$X_{CO}(t)$, $X_{HO}(t)$	$X_{CO}(t) = -9.38380 + 97.98516 X_{HO}(t) - 116.50908 X_{HO}^2(t) +$ $+63.03937 X_{HO}^3(t)$
$X_G(t)$, $X_{CO}(t)$	$X_G(t) = 0.03803 + 0.02687 X_{CO}(t)$
$X_G(t)$, $X_{HO}(t)$	$X_G(t) = 0.12943 + 0.79407 X_{HO}(t)$
$X_{HO}(t)$, $X_{CO}(t)$	$X_{HO}(t) = -3.80625 + 0.87743 X_{CO}(t) - 0.06879 X_{CO}^2(t) +$ $+0.00240 X_{CO}^3(t) - 0.00003 X_{CO}^4(t)$
$X_{HO}(t)$, $X_G(t)$	$X_{HO}(t) = -0.19987 + 2.37713 X_G(t) - 2.98411 X_G^2(t) +$ $+1.89758 X_G^3(t)$

Some remarks about the results reported in **Table 2**:

- the fact that the degree of the "best" polynomial approximation is greater than 1 in a significant percentage of the considered cases confirms the presence of non–linearities in some of the investigated bivariate dependence relationships;
- with specific regard to the fifth polynomial approximation, the fact that the coefficients associated to the highest powers of $X_{CO}(t)$ are evidently close to 0, $i.e$ the fact that their "explanatory contributions" are probably negligible, $i.e$ the fact that the degree of the approximating polynomial is probably unnecessarily high, can be interpreted as a symptom of the need that the validation procedure we propose and use here has to be probably a little bit refined.

Finally, at the end of this section we utilize all the polynomial approximations reported in **Table 2** applying each of them to the proper data subset D_F. By so doing, we provide an out–of–sample visual check (see **Fig. 1** to **Fig. 3**) of the goodness of our three–steps methodology.
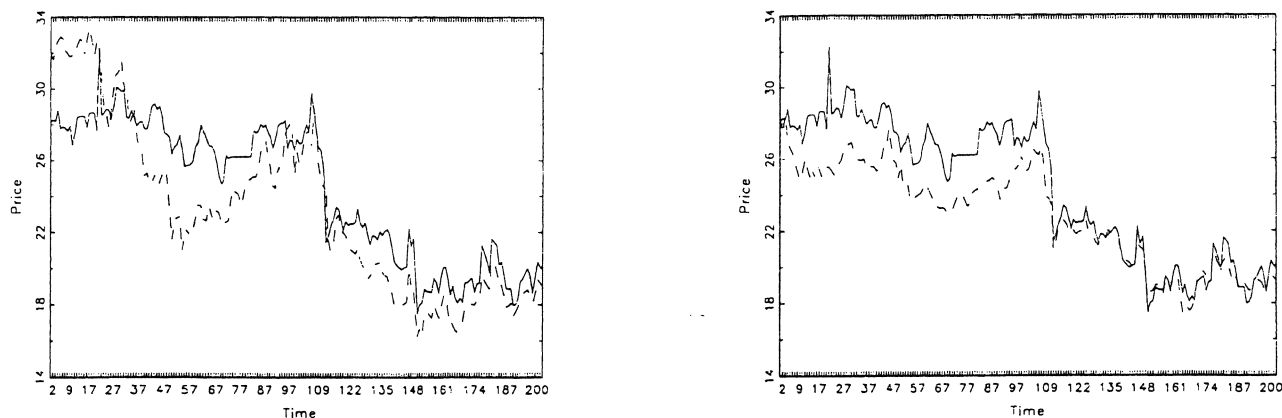
Fig. 1. In both the graphs, the continuous uneven line represents the behaviour of $X_{CO}(t)$ in the out–of–sample data subset D_F. In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{CO}(t)$ in terms of $X_G(t)$, i.e. $X_{CO}(t) = 1.68731 + 31.36198X_G(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{CO}(t)$ in terms of $X_{HO}(t)$, i.e. $X_{CO}(t) = -9.38380 + 97.98516X_{HO}(t) - 116.50908X^2_{HO}(t) + 63.03937X^3_{HO}(t)$.
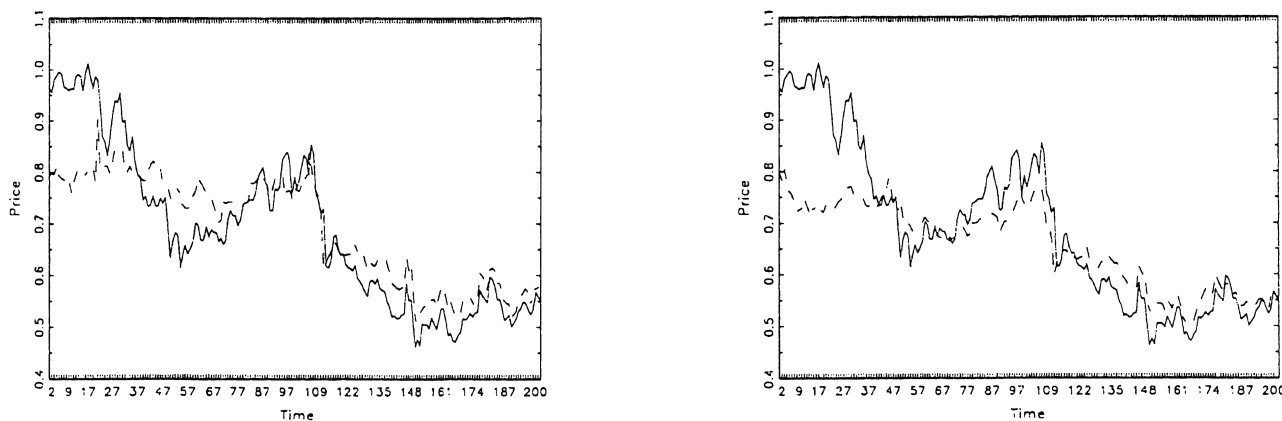

Fig. 2. In both the graph, the continuous uneven line represents the behaviour of $X_G(t)$ in the out–of–sample data subset D_F. In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_G(t)$ in terms of $X_{CO}(t)$, i.e. $X_G(t) = 0.03803 + 0.02687X_{CO}(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_G(t)$ in terms of $X_{HO}(t)$, i.e. $X_G(t) = 0.12943 + 0.79407X_{HO}(t)$.
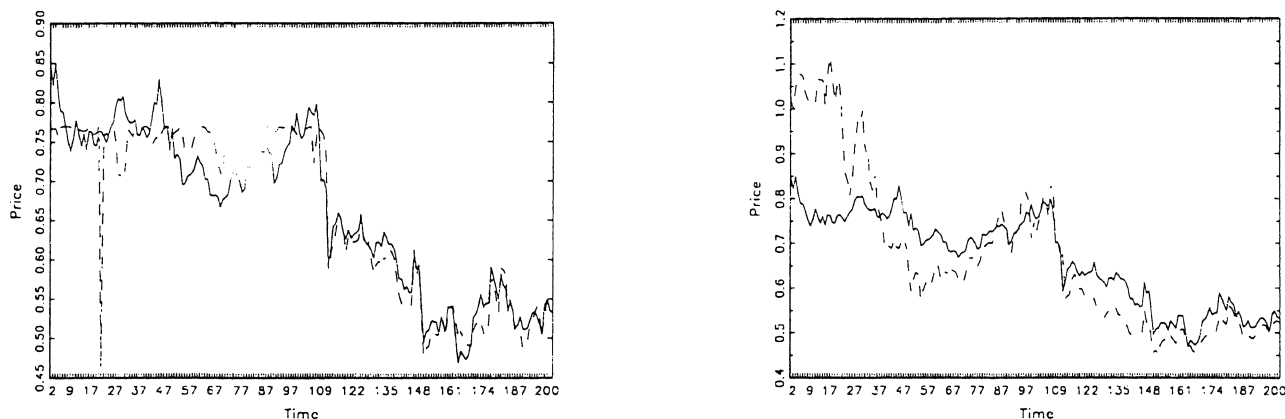
Fig. 3. In both the graph, the continuous uneven line represents the behaviour of $X_{HO}(t)$ in the out–of–sample data subset D_F. In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{HO}(t)$ in terms of $X_{CO}(t)$, *i.e.* $X_{HO}(t) = -3.80625 + 0.87743X_{CO}(t) - 0.06879X_{CO}^2(t) + 0.00240X_{CO}^3(t) - 0.00003X_{CO}^4(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{HO}(t)$ in terms of $X_G(t)$, *i.e.* $X_{HO}(t) = -0.19987 + 2.37713X_G(t) - 2.98411X_G^2(t) + 1.89758X_G^3(t)$.

Notice that, although in the graph on the right of **Fig. 3** the polynomial approximation of $X_{HO}(t)$ in D_F is generated by the approximating polynomial whose degree is probably unnecessarily high, only the estimation $\widehat{X}_{HO}(21)$ is evidently poor. We can interpret it as an indication of the robustness of our three-steps methodology.

5 Final remarks and open items

In this last section we synthetically present a few remarks concerning possible lines of investigation for future improvements and developments of the three-steps methodology we propose and use here:

- firstly, recalling that the degree of the fifth approximating polynomial reported in **Table 2** is probably unnecessarily high, surely all the validation procedure (determination of the validation data subset, specification of the validation criterion, ...) needs to be carefully verified by means of further applications of our three-steps methodology, and, on the basis of the information forecoming from such applications, it possibly needs to be properly refined;

- secondly, recalling that in subsection **4.2** we provide an out–of–sample check which is only visual, it is surely suitable to develop it in a more formal way

(like, for instance, the one given by a set of proper indices) in order to get from it more objective validation information;

— finally, we put in evidence that our three–steps methodology offers opportunities for possible generalizations. In fact, our investigating approach can be developed in order to analyze, beyond time no–lagged bivariate dependence relationships like $X_1(t) = f(X_2(t)) + \varepsilon(t)$, also time lagged bivariate dependence relationships like $X_1(t) = f(X_2(t), X_2(t-1), ..., X_2(t-N)) + \varepsilon(t)$, with $N \in \mathbb{N}^0$, time no–lagged multivariate dependence relationships like, for instance, $X_1(t) = f(X_2(t), X_3(t), ..., X_I(t)) + \varepsilon(t)$, with $I \in \mathbb{N}^0$, and time lagged multivariate dependence relationships like, for instance, $X_1(t) = f(X_2(t), X_2(t-1), ..., X_2(t-N_2), X_3(t), X_3(t-1), ..., X_3(t-N_3), ..., X_I(t), X_I(t-1), ..., X_I(t-N_I)) + \varepsilon(t)$, with $I, N_1, ..., N_I \in \mathbb{N}^0$.

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