ASSOCIAZIONE PER LA MATEMATICA APPLICATA
ALLE SCIENZE ECONOMICHE E SOCIALI

# ATTI DEL DICIOTTESIMO CONVEGNO A.M.A.S.E.S.

## Modena, 5-7 Settembre 1994

Ha contribuito alla pubblicazione del Volume
il Consiglio Nazionale delle Ricerche

PITAGORA EDITRICE BOLOGNA

Finito di stampare nel mese di Agosto 1994
presso le Officine Grafiche TECNOPRINT S.N.C.
Via del Legatore 3, Bologna.

ATTI DEL DICIOTTESIMO CONVEGNO A.M.A.S.E.S.
MODENA, 5-7 SETTEMBRE 1994



# SELECTING MEAN-VARIANCE PORTFOLIO BY NON-LINEAR MIXED INTEGER PROGRAMMING METHODS

**M. ANDRAMONOV**
Department of Applied Mathematics and Cybernetics
University of Kazan (Russia)

**M. CORAZZA**
Department of Quantitative Methods
University of Brescia (Italy)

Abstract - In this paper we propose a model for mean-variance portfolio selection in form of a non-linear mixed integer programming problem. We take into account the transaction costs, taxes and limited divisibility of the stock. For finding the optimal solution the combination of sub-gradient methods and branch and bound ones is used.

Keywords - Mean-Variance Analysis, Portfolio Selection Model, Transaction Costs, Taxes, Limited Divisibility of the Assets, Non-Linear Mixed Integer Programming, Sub-Gradient Methods, Branch and Bound Methods.

## 1. Introduction

The classical approach to the selection of mean-variance portfolio allows the distribution of a given capital amount (which is usually assumed as equal to one) between n different assets (among which at most one is riskless) in order to minimize the total risk of portfolio, measured by its variance, provided the final rate of interest.

This approach is realized by solving quadratic programming problem with linear constraints. It should be noted that in order to allow the easy solution of the optimization problem, in this approach some most simplifying hypotheses are supposed to be true:

(1.1) unlimited possibility of short sales of the assets;

(1.2) infinite divisibility of the assets;

(1.3) absence of transaction costs;

(1.4) absence of taxes.

The need to specify in more realistic way the portfolio selection approach encouraged many authors to pose the problem in corrected form, imposing the hypotheses weaker than (1.1) - (1.4).

It is of particular interest to weaken the second hypothesis, that is, to formulate a problem in the terms of the quantities of lots of assets, supposing the restrictions on the minimal number of lots to be acquired.The optimization problem,posed in such a way, has been recently studied in **Avella** (1990), **Canestrelli** and **Corazza** (1992) and **Corazza** (1991), where different branch and bound (B&B) methods of its solution have been proposed and also in **Consiglio** (1993) by using the methods of simulated annealing.

In the present paper we propose our version of the problem and an algorithm for finding optimal solution, the four hypotheses mentioned above being wea-

kened.
In particular, in section 2. a mathematical model for the selection of mean-variance portfolio is proposed, which does not require any of four hypotheses. In section 3. the algorithm for its solution is described and some theoretical results, concerning its convergence,are proved. In the section 4. we give a simple numerical example, illustrating the solution procedure and in section 5. some considerations and concluding remarks are given.

## 2. The mathematical model.

In this section we formulate a mathematical model of portfolio selection, assuming that four hypotheses, mentioned in section 1,are weakened. In particular,

(2.1) we impose the non-negativity restrictions on the number of lots to be acquired in order to avoid the possibility of short sales,which concerns only some particular categories of investors.

(2.2) the model is formulated in the terms of numbers of lots of assets, instead of percentages of capital,the presence of minimal number of lots on sale is assumed, because of the necessity for the investors to operate with lots, consisting of fixed number of assets. In particular, we pose our problem as a problem of non-linear mixed integer programming, which allows to consider the requirement to buy only some fixed quantities of lots (or do not buy nothing).

(2.3) a non-linear constraint is added, which corresponds to the transaction costs.

(2.4) another non-linear constraint is included into the model for taking some form of taxation in account.

Given these premises, the mathematical model for portfolio selection is a mixed integer programming problem:

$$(2.1)\quad \begin{cases} \min & (x'Vx) \\ \text{s.t.} & (LP_t x)'r \geq \pi C \\ & (LP_t x)'e = (1-\alpha-\beta)C \\ & f_1(x) \leq \alpha C \\ & f_2(x) \leq \beta C \\ & x \geq 0 \\ & \bar{x} \in \mathbb{N}, \end{cases}$$

where

x - an unknown vector ((n+1)x1) of the numbers of lots to be acquired (for each asset);

V - a known nxn matrix of variance and covariance of interest rates of assets, it is simmetric and its elements are of the form:

$$\sigma(i,j) = s(i)s(j)l(i)l(j),$$

where s(i) is a mean quadratic deviation of interest rate for *i*th asset , l(i) is a minimal number of lots of *i*th asset , which can be acquired;

L - known diagonal matrix,

$$l_{ii} = l(i)\ ,\ l_{ij} = 0\ i\neq j;$$

$P_t$-the matrix of the prices of assets at time t:

$$P_{ii}(t) = p_i(t),\ P_{ij} = 0\ i\neq j;$$

r - known vector of mean values of interest rates of assets;

$\pi$ - the rate of interest, which an investor expects to obtain from his portfolio. It is necessary that

$$(1-\alpha-\beta)\pi_{min} \le \pi \le (1-\alpha-\beta)\pi_{max};$$

C - initial capital of the investor;

e - vector ((n+1)x1), all elements of which are equal to 1;

$\alpha$, $\beta$ - the scalar parameters, indicating, respectively, the maximal percentages of capital to be spent for the transaction costs and taxes. The investor must choose

$$\alpha > 0 \ , \ \beta > 0, \ \alpha+\beta < 1;$$

$f_1(x)$ - the function of transaction costs for a given portfolio x;

$f_2(x)$ - the taxes to be paid for the acquist of the portfolio x;

I - set of the indices, corresponding to discrete assets, $I \subseteq \{0,1,...,n\}$;

0- the vector ((n+1)x1) with zero components.

The model in this form does not require the hypotheses like 1-4. However, we should note that if all the assets are discrete, then often it is impossible to distribute all the capital C, that is to satisfy the constraint

$$(LP_t x)'e = (1-\alpha-\beta)C$$

and the feasible set would be empty in such case.

In order to avoid this undesirable situation we suppose that at least one of the assets is real (infinitely divisible). This asset can be risky or it can be a riskless bond (it is not important).

## 3. The algorithm for solving problem (2.1).

In order to find an optimal solution, we suppose that the functions $f_1, f_2$ are strictly pseudoconvex. Recall that a function $f(x)$, defined on the euclidean space $E_n$, is called strictly pseudoconvex, if at any point $y \in E_n$ the inequality $f(x)<f(y)$ implies

$$(f'(y), x-y) < 0$$

The algorithm can be generalized for non-differentiable functions, but then it is more difficult to implement it. Practically, we use the combination of the cutting plane methods [**Kelley,Demyanov,Vasilyev**] and branch and bound [**Nemhauser,Wolsey**].

The solution procedure consists of two stages. First, we find an initial point $x_0$, satisfying the following constraints:

$$(3.1) \quad \begin{cases} (LP_t x_0)'r \geq \pi C \\ (LP_t x_0)'e = (1-\alpha-\beta)C \\ x_0 \geq 0, \; x_{0i} \in \mathbb{N} \;\; \forall i \in I, \; x_{0i} \in \mathbb{R} \;\; \forall i \notin I, \\ x = (x_{00}, x_{01}, x_{02}, .., x_{0n}). \end{cases}$$

Usually, there exists a feasible point, in which only two components are different from 0, one of which corresponds to risky asset and another to a bond.

In this case we obtain the following conditions:

$$P_k L_k x_{0k} + P_0 L_0 x_{00} = (1-\alpha-\beta)C$$
$$r_k P_k L_k x_{0k} + r_0 P_0 L_0 x_{00} \geq \pi C$$
$$x_{0k} \in \mathbb{N}, \; x_{00} \in \mathbb{R}, \; x_{0k} \geq 0, \; x_{00} \geq 0.$$

As typically $r_0 < r_k$, it is reasonable to take

$$x_{0k} = \left[ \frac{(1-\alpha-\beta)C}{P_k L_k} \right],$$

$$x_{00} = \frac{(1-\alpha-\beta)C - P_k L_k x_{0k}}{P_0 L_0},$$

$x_{0i}=0$, $i \neq k$, $i \neq 0$. The brackets mean the maximal integer, which does not exceed the expression inside. If for some k such point satisfies (3.1), we take it as initial feasible point.

However, it can happen that no one of such points satisfies (3.1) and a solution still exists. We think that this situation has little probability, which can be illustrated by some examples. In this case we propose to solve a knapsack type problem:

$$\begin{cases} \max \quad rPLy \\ \qquad ePLy = (1-\alpha-\beta)C \\ \qquad y \geq 0 \\ \qquad y_i \in \mathbb{N} \ \forall i \in I, \ y_i \in \mathbb{R} \ \forall i \notin I, \ \{0\} \in I. \end{cases}$$

For solving it the standard methods can be used, see, for example [**Martello, Toth**].

If in the optimal solution $y^*$ of (3.1)

$$rPLy^* \geq \pi C$$

then we can set $x_0 = y^*$. Otherwise the feasible set of initial problem is empty and at least one of the parameters $(\pi, \alpha, \beta)$ is to be revised.

Suppose now that $x_0$ has been found. The second stage of the algorithm is as follows.

<u>Step 0</u> Find an initial point $x_0$. Let k=0. If $f_1(x_0) \leq$ $\leq \alpha C$, $f_2(x_0) \leq \beta C$ then let

$$x^* = x_0; \quad f^* = f(x_0).$$

Else $f^* = +\infty$.

Step 1 If $f_1(x_0) > \alpha C$ then $g_k = f_1'(x_k)$ and go to step 4.

Step 2 If $f_2(x_0) > \beta C$ then $g_k = f_2'(x_k)$ and go to step 4.

Step 3 $g_k = 2Vx_k$ .

Step 4 Find a solution of the system

$$\begin{cases} rPLx \geq \pi C \\ ePLx = (1-\alpha-\beta)C \\ (g_i, x-x_i) < 0 \quad i=0,1,2,\dots,k \\ x_{ij} \geq 0,\ x_{ij} \in \mathbb{N}\ (j \in I),\ x_{ij} \in \mathbb{R}\ (j \notin I), \\ x_i = (x_{i0}, x_{i1}, \dots, x_{in}). \end{cases}$$

Denote it z. If there are no solutions, go to step 6.

Step 5 If z is feasible, that is

$$f_1(z) \leq \alpha C, \quad f_2(z) \leq bC$$

and if also $f(z) < f^*$ then let $x^* = z$, $f^* = f(x^*)$.

Let $k := k+1$ and go to step 1 with $x_k = z$.

Step 6 If $f^* = +\infty$ then STOP, $x^*$ is an optimal solution. Else STOP, the feasible set is empty.

Note that at step 4 we should find a solution of the system of linear inequalities and equation with atmost n integer and at least one real variable. It can be found, if we use any known branch and bound techniques (see [**Nemhauser, Wolsey**] and the references therein).

Theorem. In a finite number of iterations the algorithm either finds an optimal solution or indicates that the feasible set is empty.

Proof. For any $i=0,1,\dots,n$ the values $P_i, L_i$ are positive, so the number of the points, satisfying

$$(3.2) \quad \begin{cases} ePLx = (1-\alpha-\beta)C \\ x \geq 0; \quad x_i \in \mathbb{N}\ \forall i \in I; \quad x_i \in \mathbb{R}\ \forall i \notin I \end{cases}$$

is finite. Any point $x_k$ satisfies the conditions (3.2) and cannot be found more then once, so the algorithm terminates in a finite number of iterations.

Suppose that at step 6 we have $f^* = +\infty$. It means that $g_i$ was always equal to $f_1'(x_i)$ or $f_2'(x_i)$ and thus all the points $x_k$ are infeasible. But the feasible set belongs to the polyhedron, defined on step 4 and thus is empty.

Finally, let $f^* \neq +\infty$. By construction, it means that $f(x^*) \leq f(x_k)$ for any k such that $x_k$ is feasible.

Then the system:

$$\begin{cases} rPLx \geq \pi C \\ ePLx = (1-\alpha-\beta)C \\ (2Vx^*, x-x^*) < 0 \\ (g_i, x-x_i) < 0 \quad i=1,2,\ldots,k \\ x\geq 0; \quad x_j \in \mathbb{N} \ \forall j \in I; \quad x_j \in \mathbb{R} \ \forall j \notin I \end{cases}$$

has no solutions. But this polyhedron contains the fesible set, intersected with the level set

$$\{ y \in E_n \mid y'Vy < x^*Vx^* \},$$

so there are no feasible solutions with the objective function less than $x^*Vx^*$. It means that $x^*$ is a minimum point. Thus the theorem is proved.

In some cases the convergence of such algorithm can be slow, so it may be necessary to use some tools for providing faster convergence. Certainly, much depends on the choice of the solution on step 4, if there are many.

Remark 1 If k becomes too large, we can throw away some equations $(g_i, x-x_i)<0$, for example, those that always were inactive during a fixed number of iterations.

Remark 2 We should choose the points, which are "deep" enough inside the feasible set. Choose a system of orthonormal vectors $u_i$, orthogonal to the vector ePL and a parameter $\lambda>0$. Then it would be wise to require that any point $x\pm \lambda u_i$ should be feasible (without the requirement that its coordinates should be integer).

Remerk 3 Most known methods do not work with strict inequalities. Thus we can write instead

$$(g_i, x-x_i) \leq -\varepsilon \quad i=1,2,...,k ,$$

where $\varepsilon>0$ is a fixed small positive number. It is to be chosen with care in order not to lose some feasible points.

Remark 4 We can use some linear objective function at step 4, for example, maximize $(g_{k-1}, x)$ in order to "go away" from the previous point.

## 4. Numerical example.

Suppose that we have two risky discrete assets and the data for the portfolio selection are the following:

$$p= (3;7); \quad l(1)=l(2)=1; \quad C=100;$$

$$r = (0.2;0.4); \quad \pi=0.25; \quad \alpha=0.1; \quad \beta=0.2$$

$$f_1(x) = 2 (\sqrt{x_1} + \sqrt{x_2})$$

$$f_2(x) = 2x_1 + 2x_2 .$$

$$V = \begin{pmatrix} 0.6 & -0.5 \\ -0.5 & 1 \end{pmatrix}.$$

Then the mathematical model can be written as the following mathematical programming problem:

$$\max \quad 0.6\, x_1^2 + x_2^2 - x_1 x_2$$

$$\text{s.t.} \quad 0.6\, x_1 + 2.8\, x_2 \geq 25$$

$$3x_1 + 7x_2 \leq 70$$

$$\sqrt{x_1} + \sqrt{x_2} \leq 5$$

$$x_1 + x_2 \leq 10$$

$$x_1, x_2 \geq 0;\ x_1, x_2 \in \mathbb{N}.$$

Obviously, we can take as an initial feasible point the vector (0;10). For the functions of costs and taxes the constraints are satisfied, so we should calculate the gradient of the objective function, which is equal to (-10;20), letting also $x^*=(0;10)$, $f^*=100$.

Then we add the inequality

$$-10x_1 + 20x_2 < 200.$$

The next feasible point, which can be taken, is (0;9). All the constraints are satisfied, so we should calculate again the gradient of the objective function, it is the vector (-9;18). The value of the objective function at this point is 81, so let $x^*=(0;9)$, $f^*=81$.

Thus the second constraint to be added is:

$$-9x_1 + 18x_2 < 162.$$

One of the possible solutions of our auxiliary linear system is (2;9). However, for this point the constraint, given by the function of the taxes, is violated and we must calculate the gradient of $f_2$, which is the vector (1;1) and add the inequality

$$x_1 + x_2 < 11.$$

The next solution is the point (1;9). All the constraints are satisfied, the value of objective function is 72.6, so $x^*=(1;9)$, $f^*=72.6$.

Adding again the inequality, corresponding to the gradient of the objective function at this point:

$$-7.8x_1 + 17x_2 < 145.2 \ ,$$

we see that the set of feasible solutions is empty.

As $f^* \neq +\infty$ , $x^* = (1;9)$ is a point of minimum of our objective function. It means that the investor should buy one lot of first asset and nine lots of second one.

The minimal possible risk,measured by the variance of portfolio, is equal to 72.8.

## 5. Final conclusions and considerations.

In respect to the mathematical model, presented in section 2. and to theoretical results from section 3. the following final conclusions and remarks can be given:

(5.1) two constraints, introduced in order to consider the presence of transaction costs and taxes do not require any particular hypotheses,it is enough to suppose the differentiability and pseudo-convexity of corresponding functions;

(5.2) for the solution of the problems with integer variables it is not necessary to elaborate complicated specific methods, one can apply the combination of well known branch and bound and cutting planetechniques, adjusting it to non-linear programs;

(5.3) the algorithm does not require that the objective function should be quadratic, it must be only differentiable and pseudo-convex;

(5.4) the method of portfolio selection provides the information about the "economic compatibility" of the values of parameters $\alpha, \beta, \pi$ , assigned by the investor, with real economic situation, in which the investor plans to act. Indeed,if the choice of these parameters is incoherent with economico - financial system , the sets of feasible solution will have little financial sense or even can be empty.

The main directions of future developments of the model and algorithm, presented in this paper, would be the following:

(5.5) it is intended to generalize the model as well as the numerical algorithm ,allowing to use the objective function and functions of costs ,which are not necessarily pseudo- or quasi-convex;

(5.6) it is important to analyse the sensitivity with respect to the minimal numbers of lots, which can be acquired. These values are exogenous for the model and prove to be a powerful instrument of financial policy of the Authorities of Exchange,being able to influence significantly the optimal portfolio selection, effectuated by the investors.

(5.7) it is necessary to make a large number of experiments with the algorithm for solving the problem of portfolio selection with costs and taxes, thus comparing it with classical portfolio selection methods. The application of our method to real problems will allow to verify its practical utility.

## 6. References

**AVELLA, P.** - Portfolio Selection: un Modello di Programmazione Mista - Intera risolto con un Algoritmo Branch and Bound, Ricerca Operativa, N. 56, 3-28, 1990.

**CANESTRELLI, E.** and **CORAZZA, M.** - Un Modello Risolutivo a Variabili Miste-Intere per la Selezione di Portafoglio in Media-Varianza, Atti del Sedicesimo Convegno A.M.A.S.E.S., Treviso, 203-218, 1992.

**CANESTRELLI, E.** and **NARDELLI, C.** - Criteri per la Selezione del Portafoglio, G. Giapicchelli Editore, Torino, 1991.

**CANESTRELLI, E.** and **TOSATO, C.** - Funzioni di Penalita' Esterna nel Problema della Selezione del Portafoglio, Quaderni del Dipartimento di Matematica Applicata ed Informatica della Universita' degli Studi di Venezia, N. 62/90, 1990.

**CASTAGNOLI, E.** and **PECCATI, L.** - Introduzione alla Selezione del Portafoglio, Cooperativa di Cultura "Lorenzo Milani", Milano, 1991.

**CORAZZA, M.** - Considerazioni sulla Applicazione di un Algoritmo di tipo "Branch and Bound" ad un Modello a Variabili Miste per la Selezione di Portafoglio in Media-Varianza, Rendiconti del Comitato per gli Studi Economici, XXIX, 63-82, 1991.

**DEMYANOV, V.F.** and **VASILIEV, L.V.** - Nondifferentiable Optimization, Springer-Verlag, Berlin, 1985.

**ELTON, E. J.** and **GRUBER, M. J.** - Modern Portfolio Theory and Investment Analysis, John Wiley Sons, New

York - Chichester - Brisbane - Toronto - Singapore, 1984.

**KELLEY, J. E.** - The Cutting Plane Method for Solving Convex Programs, Journal of the SIAM, 8, 703-712, 1960.

**LUENBERGER, D. G.** - Introduction to Linear and Non-linear Programming, Addison-Wesley, Menlo Park, California, 1984.

**MARKOWITZ, H. M.** - Mean - Variance Analysis in Portfolio Choice and Capital Markets, Basil Blackwell, Oxford - Cambridge, 1989.

**MARKOWITZ, H. M.** - Portfolio Selection. Efficient Diversification of Investments, Basil Blackwell, Oxford - Cambridge, 1991.

**MARTELLO, S.** and **TOTH, P.** - Knapsack Problems. Algorithms and Computer Implementations, Wiley, New York, 1990.

**NEMHAUSER, G. L.** and **WOLSEY, L. A.** - Integer and Combinatorial Optimization, Wiley, New York, 1988.

**OMPRAKASH, K. G.** and **RAVIDAN, A.** - Branch and Bound Experiments in Convex Nonlinear Integer Programming, Management Science, Vol. 31, N. 12, 1533-1546, 1985.

**ROSSI, F. A.** and **TREGLIA, B.** - La Selezione del Portafoglio secondo il Criterio Valore Atteso-Varianza, Libreria Universitaria Editrice, Verona, 1994.

**SZEGO, G. P.** - Portfolio Theory. With Application to Bank Asset Management, Academic Press, New York - San Francisco - London, 1980.