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Mixed-integer non-linear programming methods for mean-variance portfolio selection

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Abstract. In standard mean-variance portfolio selection, several simplifying hypotheses are usually assumed. On the contrary, in this paper we weaken some of the most common of them, and we propose a problem of portfolio selection in which the following realistic aspects are taken into account: the impossibility of short sale of the assets; their not infinite divisibility; and the presence of transaction costs and taxation. The mathematical formulation of this selection problem is given in term of a mixed-integer non-linear programming one. In order to find its optimal solution (if any), we developed a two-stage solving algorithm (which is based on the branch and bound method, on the cutting plane one, and on the sub-gradient method), and we prove its convergence in a finite number of iterations.

KEYWORDS: mean-variance portfolio selection, impossibility of short sale, not infinite divisibility of the assets, transaction costs, taxation, mixed-integer non-linear programming, branch and bound method, cutting plane method, sub-gradient method.

J.E.L. CLASSIFICATION: C61, C63, G11.
M.S.C. CLASSIFICATION: 90C11, 90C20, 49M37.

1 Introduction

It is well known that the classical approach to the mean-variance portfolio selection allows the distribution of a given initial capital (which is usually assumed equal to 1) among n different assets (of which at most one is riskless) in order to minimize the total risk of the portfolio (which is measured by its variance) provided a proper rate of return the investor whishes to obtain (see, for standard and advanced introductions, [Szegö, 1980], [Elton *et al.*, 1984], [Markowitz, 1989] and [Markowitz, 1991]). Generally, also in order to allow an easy solution of the corresponding programming problem, in such an approach several simplifying (but unrealistic) hypotheses are usually assumed. In particular, the most common are:

1.1 the possibility of short sale of the assets;
1.2 their infinite divisibility;
1.3 the absence of transaction costs;
1.4 the absence of taxation.

The need to specify in a more realistic (and also operative) way the portfolio selection approach encouraged many Authors to consider the corresponding problem in a more suitably general form, taking into account hypotheses weaker than 1.1 to 1.4. In particular, there already exist some portfolio selection models in which the hypothesis 1.2 is weakened and the corresponding programming problem is formulated in terms of lots of assets[1] (see, for example, [Avella, 1990], [Corazza, 1991], [Canestrelli *et al.*, 1992] and [Corazza *et al.*, 1999]).

In this paper we propose a portfolio selection problem, and solve it, in which all the above listed hypotheses are weakened. In details: in section **2** we present the mathematical programming problem related to the considered mean-variance portfolio selection one; in section **3** we propose our two-stage solving algorithm and we give a theoretical result on its convergence; in section **4** we present a simple numerical example; and in section **5** we give some final remarks.

2 The model

In this section we present our formulation of the mathematical programming problem corresponding to the considered portfolio selection model. In particular:

- we formulate the model in terms of number of lots instead of percentages of capital (weakening of the hypothesis 1.2); in details, we pose our programming problem as a mixed-integer non-linear one;
- we consider non-negativity constraints on the number of lots, so avoiding the possibility of short sales (weakening of the hypothesis 1.1);
- we take into account a first non-linear constraint associated to the transaction costs (weakening the of hypothesis 1.3);

[1] A lot is the minimal prefixed (integer) quantity of a given asset which is possible to buy or to sell.

– we take into account a second non-linear constraint associated to the taxation (weakening the of hypothesis 1.4).

Given these premises, we mathematically formulate the considered portfolio selection problem as the following mixed-integer non-linear programming one:

$$\begin{array}{ll} \min & f(x) = x'Vx \\ \text{s.t.} & (LPx)'r \geq \pi C \\ & f_1(x) \leq \alpha C \\ & f_2(x) \leq \beta C \\ & (LPx)'e = (1-\alpha-\beta)C \\ & x_i \geq 0 \ \forall \ i \in \{1,\ldots,n\} \\ & x_j \in \mathbb{N} \ \forall \ j \in I \end{array} \tag{1}$$

where
$x \in \mathbb{R}^n$ is the unknown vector of the quantities of lots to buy;
$V \in \mathbb{R}^n \times \mathbb{R}^n$ is the known matrix of variance and covariance of the lot returns; its elements have the form

$$V(i,j) = [\sigma(i)l(i)][\sigma(j)l(j)]\rho(i,j),$$

in which $\sigma(v)$ is the standard deviation of the return of the v-th asset, $l(v)$ is the prefixed number of assets constituting the v-th lot, and $\rho(v,w)$ is the correlation coefficient between the return of the v-th asset and the return of the w-th one;
$L \in \mathbb{N}^n \times \mathbb{N}^n$ is the known diagonal matrix of the number of assets constituting each lots; its elements have the form

$$L(i,j) = \begin{cases} l(i) \in \mathbb{N} \setminus \{0\} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases};$$

$P \in (\mathbb{R}^+ \cup \{0\})^n \times (\mathbb{R}^+ \cup \{0\})^n$ is the known diagonal matrix of the asset prices at the current time; its elements have the form

$$P(i,j) = \begin{cases} p(i) \in \mathbb{R}^+ \setminus \{0\} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases};$$

$r \in \mathbb{R}^n$ is the known vector of the mean values of the asset returns;
π is the known return rate which the investor wishes to obtain from the portfolio; notice that, because of the presence of the non-negativity constraints, it is necessary to impose

$$(1-\alpha-\beta)r_{\min} \leq \pi \leq (1-\alpha-\beta)r_{\max},$$

in which $r_{\min} = \min\{r_i, \ i = 1, \ldots, n\}$ and $r_{\max} = \max\{r_i, \ i = 1, \ldots, n\}$;
C is the initial capital available to the investor;
$f_1(\cdot)$: $\mathbb{R}^n \to \mathbb{R}$ is the non-linear function associated to the transaction costs;
$f_2(\cdot)$: $\mathbb{R}^n \to \mathbb{R}$ is the non-linear function associated to the taxation;
α and β are known scalar parameters indicating the maximal percentages of the initial capital which is possible to spent, respectively, for the transaction costs and for the taxation; in particular, the investor has to choose α and β such that

$$\alpha > 0, \ \beta > 0 \ \text{and} \ \alpha + \beta < 1;$$[2]

$e \in \mathbb{R}^n$ is a vector whose elements are all equal to 1;
$\overline{x} \in \mathbb{N}^m$, with $m \leq n$, is an unknown vector whose elements belong to x.[3]

Notice that, in general, if m were equal to n, then it could be often impossible to distribute all the capital $(1 - \alpha - \beta)C$ among the assets, that is (in equivalent terms) to satisfies the constraints

$$(LPx)'e = (1 - \alpha - \beta)C;$$

in such a case the feasible set should be empty.

In order to avoid this undesirable situation, we suggest, among the others, the two following possible solutions (to use alternatively or jointly):

- substituting the considered constraint with its "inequality version" $(LPx)'e \leq (1 - \alpha - \beta)C$;
- imposing the existence of at least an infinitely divisible asset.

3 The algorithm for solving problem (1)

In order to find an optimal solution of the mathematical programming problem (1), on the following of this paper we assume:

[2] Notice that without the third restriction, i.e. without $\alpha + \beta < 1$, the considered problem could become meaningless because $f_1(x) + f_2(x)$, which is equal to $(\alpha + \beta)C$, could be equal to or greater than the initial capital available to the investor.

[3] On the following, we denote by I the set of the indexes corresponding to the not infinitely divisible assets.

- that the function $f_1(\cdot)$ and $f_2(\cdot)$ are both strictly pseudo-convex[4] (notice that the solving algorithm we propose in this section could be also generalized for non-differentiable $f_1(\cdot)$ and $f_2(\cdot)$, but that the corresponding practical implementation should be more difficult than the one we present here);
- that all but at most one assets are not infinitely divisible; in particular, if such an infinitely asset there exists, then we let $\bar{\imath} = 2$ and $I = \{2, \ldots, n\}$; else we let $\bar{\imath} = 1$ and $I = \{1, \ldots, n\}$.

To realize our algorithm we use a suitable combination of the branch and bound method (see [Omprakash *et al.*, 1985] and)[Nemhauser et al., 1988], of the cutting plane one (see [Kelley, 1960] and [Demyanov et al., 1985]) and of the sub-gradient method. The final solving procedure consists of two stages.

3.1 First stage

At first we find an initial point x_0 satisfying the system of constraints which follows:

$$\begin{cases} (LPx)'r \geq \pi C \\ (LPx)'e = (1-\alpha-\beta)C \ \ (\text{or } (LPx)'e \leq (1-\alpha-\beta)C) \\ x(i) \geq 0 \ \forall \ i \in \{1, \ldots, n\} \\ x(i) \in \mathbb{N} \ \forall \ i \in I \end{cases} . \tag{2}$$

In order to make easy the finding of this initial point, we assume that an infinitely divisible asset exists, namely $x_0(1)$. Now, it is usually easy enough to find an initial point characterized by only two components which differ from 0. In fact, if there exists a $k \in \{2, \ldots, n\}$ such

[4] Recall that a real valued function $f(\cdot)$ on the Euclidean space $\mathbf{E}_n$ is defined strictly pseudo-convex if at any point $y \in \mathbf{E}_n$ the inequality $f(x) < f(y)$, with $x \in \mathbf{E}_n$, implies the inequality $\langle \nabla(f(y)), x - y \rangle < 0$, in which $\langle \cdot, \cdot \rangle$ indicates the inner product operator.

that

$$\begin{cases} p(1)l(1)x_0(1)r(1) + p(k)l(k)x_0(k)r(k) \geq \pi C \\ p(1)l(1)x_0(1) + p(k)l(k)x_0(k) = (1-\alpha-\beta)C \;\; (\text{or } p(1)l(1)x_0(1) + \\ \qquad\qquad\qquad + p(k)l(k)x_0(k) \leq (1-\alpha-\beta)C) \\ x_0(1) > 0 \\ x_0(k) > 0 \\ x_0(k) \in \mathbb{N} \end{cases}, \tag{3}$$

then we obtain such an initial point in which

$$x_0(k) = \left\lfloor \frac{(1-\alpha-\beta)C}{l(k)p(k)} \right\rfloor,$$
$$x_0(1) = \frac{(1-\alpha-\beta)C - l(k)p(k)x_0(k)}{l(0)p(0)},$$
$$x_0(i) = 0 \;\forall\; i \in \{2, \dots, n\} \setminus \{k\}$$

where

$\lfloor\cdot\rfloor$ means the maximal integer value which does not exceed the value of the expression inside.

Of course, it could happen that the system (3) is not satisfied for any $k \in \{2, \dots, n\}$ but that an initial point satisfying the system (2) still exists. Although we conjecture that this occurence is highly unlikely, in order to find the related initial point we propose to solve the knapsack type problem which follows (by using standard methods like the ones presented in [Martello *et al.*, 1990]):

$$\begin{array}{ll} \max & (LPy)'r \\ \text{s.t.} & (LPy)'e = (1-\alpha-\beta)C \;\; (\text{or } (LPy)'e \leq (1-\alpha-\beta)C) \\ & y_i \geq 0 \;\forall\; i \in \{1, \dots, n\} \\ & y_j \in \mathbb{N} \;\forall\; j \in I \end{array}.$$

If the optimal solution y^* of this problem is such that $(LPy^*)'r \geq \pi C$, then we let $x_0 = y^*$; else it results that the corresponding feasible set is empty and at least one of the (financial) parameters π, α and β has to be properly revised.

3.2 Second stage

In the second stage, given an initial point x_0, we find an optimal solution of the problem (1) by using the following algorithm we originally developed:

step 0: let $h = 0$;
step 1: if $f_1(x_h) \leq \alpha C$ and $f_2(x_h) \leq \beta C$, then let $x^* = x_h$, $f^* = f(x_h)$ and go to step 4;
step 2: if $f_1(x_h) > \alpha C$, then let $g_h = \nabla(f_1(x_h))$ and go to step 5;
step 3: if $f_2(x_h) > \beta C$, then let $g_h = \nabla(f_2(x_h))$ and go to step 5;
step 4: let $g_h = \nabla(f(x_h)) = 2Vx_h$;
step 5: find a solution of the auxiliary system

$$\begin{cases} \langle g_0, x - x_0 \rangle < 0 \\ \vdots \\ \langle g_h, x - x_h \rangle < 0 \\ x_h(j) \geq 0 \ \forall \ j \in \{1, \ldots, n\} 11 x_h(j) \in \mathbb{N} \ \forall \ j \in I \end{cases} ;$$

if such a solution there exists, then denote it z; else go to step 7;
step 6: if $f(z) < f^*$, then let $h = h + 1$, $x_h = z$ and go to step 1;
step 7: if $f^* \neq +\infty$, then stop and release x^* and f^*; else stop and indicate that the feasible set is empty.

Notice that in order to solve the system at step 5, the integrity constraints are "managed" by standard branch and bound techniques, like the ones in [Nemhauser *et al.*, 1988] and in the references therein.

As far it is concerned the solving algorithm proposed in this second stage, we give and prove the following theoretical result.

Theorem If all the assets but al most one are not infinitely divisible, then in a finite number of iterations the algorithm either finds an optimal solution or indicates that the feasible set is empty.
Proof For any $i = \bar{i}, \ldots, n$, both $l(i)$ and $p(i)$ are positive and finite, so the number $\bar{l}$ of initial points satisfying the system

$$\begin{cases} (LPx)'e = (1 - \alpha - \beta)C \ \ (\text{or } (LPx)'e \leq (1 - \alpha - \beta)C) \\ x(i) \geq 0 \ \forall \ i \in \{1, \ldots, n\} \\ x(j) \in \mathbb{N} \ \forall \ j \in I \end{cases} \tag{4}$$

is finite.[5] In particular:

[5] Do not confuse the number $\bar{l}$ of initial points satisfying the system (4) with the counter h used in the algorithm.

- if all the assets are not infinitely divisible, i.e. $\bar{\imath} = 1$, then each $x(i)$, with $i = 1, \ldots, n$, can assume value in a set, $X_i = \{0, 1, ..., \lfloor(1-\alpha-\beta)C/l(i)p(i)\rfloor\}$, whose cardinality is $\lfloor(1-\alpha-\beta)C/l(i)p(i)\rfloor + 1 < +\infty$; so, the number of all their possible combinations (in the sense of the cardinality of the set $X_1 \times X_2 \times \cdots \times X_n$) is finite;[6]
- if all the assets but one are not infinitely divisible, i.e. $\bar{\imath} = 2$, then for all the finitely divisible assets is true what proved in the previous point; because of that, also $x(1)$, which is residually equal to $\left[(1-\alpha-\beta)C - \sum_{i=2}^{n} l(i)p(i)x_0(i)\right]/l(1)p(1)$, can assume value in a set whose cardinality is finite; so, again, the number of all the possible combination of $x(i)$, with $i = 1, \ldots, n$, is finite;
- if the number of assets which are infinitely divisible is $2 \leq \bar{\imath} < n$, then the number of initial points satisfying the "residual" system

$$\begin{cases} \sum_{i=1}^{\bar{\imath}} l(i)p(i)x(i) = (1-\alpha-\beta)C - \sum_{i=\bar{\imath}+1}^{n} l(i)p(i)x_0(i) \\ x(j) \geq 0 \ \forall \ j \in \{1, \ldots, \bar{\imath}\} \end{cases}$$

is not finite when such a system admits not trivial solutions, that is when $\left(x^*(1), \ldots, x^*(\bar{\imath})\right) \neq (0, \ldots, 0)$, that is when $(1-\alpha-\beta)C - \sum_{i=\bar{\imath}+1}^{n} l(i)p(i)x_0(i) > 0$.

Each of the initial points satisfying the system (4) can not be found than once; so, the algorithm can iterate at most $\bar{l} < +\infty$ times.
Now, we prove that the algorithm finds an optimal solution in a finite number of iterations when such a solution exists. If for some of the initial points $x_{\bar{p}}$, with $\bar{p} \in \{1, \ldots, \bar{l} < +\infty\}$, f^* differs from $+\infty$ at step 7, then by construction $f^* < f(x_i)$ for all x_i such that $i \in \{0, \ldots, h\}$ and x_i is feasible; because of that, the system

$$\begin{cases} (LPx)'r \geq \pi C \\ (LPx_0)'e = (1-\alpha-\beta)C \ \ (\text{or } (LPx_0)'e \leq (1-\alpha-\beta)C) \\ \langle 2Vx^*, x - x^* \rangle < 0 \\ \langle g_0, x - x_0 \rangle < 0 \\ \vdots \\ \langle g_h, x - x_h \rangle < 0 \\ x(i) \geq 0 \ \forall \ i \in \{1, \ldots, n\} \\ x(j) \in \mathbb{N} \ \forall \ j \in I \end{cases} \tag{5}$$

[6] Notice that, in general, $\bar{l}$ is less than or equal to this number of combinations.

does not admit solutions. In particular, again by construction, this system specifies a polyhedron to which belongs the intersection between the feasible set of the starting mathematical programming problem and the level set $\{y : y'Vy < x^{*\prime}Vx^*\}$; so, because the system (5) does not admit solution, then the considered intersection is empty and x^* is an optimal solution.

Finally, we prove that the algorithm indicates in a finite number of iterations that the feasible set is empty when the starting mathematical programming problem does not admit solution. If for all the initial points x_i, with $i = 1, \dots, \bar{l} < +\infty$, f^* is equal to $+\infty$ at step 7, then each $g_i(x_i)$, with $i = 1, \dots, \bar{l}$, is always equal to $\nabla f_1(x_i)$ or to $\nabla f_2(x_i)$, that is all the considered x_i are unfeasible. So, the system (5) does not admit solution for any initial point x_i, with $i = 1, \dots, \bar{l}$; consequently, the polyhedron implied by this system is an empty set, from which forecomes that also the related feasible set of the starting mathematical programming problem is empty. ■

Notice that in some cases the convergence of the proposed algorithm can be slow; because of that, in these cases it could be useful to utilize some techniques able to provide faster convergence (for instance, much depends on the choice of the solution of the system in step 5 when they are many).

Moreover, for the ordinary utilization of the considered algorithm it can be useful to take into account the following suggestions:

- if h becomes large, then one could throw away from the system in step 5 some of the inequalities $\langle g_i, x - x_i \rangle < 0$, with $i = 0, \dots, h$, choosing, for example, the ones that results inactive during a prefixed number of iterations;
- among all the initial points satisfying the system (4), one should have to choose as starting one a point which is "sufficiently deep" in the feasible set;
- recalling that several optimization methods do not work with strict inequalities, in order to solve the system in step 5 by using such methods one has to replace each $\langle g_i, x - x_i \rangle < 0$ with $\langle g_i, x - x_i \rangle \leq \epsilon$ for all $i \in \{1, \dots, h\}$, where $\epsilon \in \mathbb{R}^+ \setminus \{0\}$ is a properly small number.

4 Numerical example

In this section we give a simple numerical example. One considers two risky assets and the following related data necessary for the portfolio selection:

$$V = \begin{pmatrix} 0.6 & -0.5 \\ -0.5 & 1.0 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; P = \begin{pmatrix} 3.0 & 0.0 \\ 0.0 & 7.0 \end{pmatrix}, \; r = \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix},$$
$$\pi = 0.25, \; C = 100, \; \alpha = 0.1, \; \beta = 0.2,$$
$$f_1(x) = \frac{200}{81}\left(\sqrt{x_1} + \sqrt{x_2}\right), \; f_2(x) = 2\left(x_1 + x_2\right),$$
$$x_1, \; x_2 \in \mathbb{N}.$$

The corresponding portfolio selection problem is given by the following mathematical programming one:

$$\begin{array}{ll} \min & f(x) = 0.6x_1^2 + x_2^2 - x_1x_2 \\ \text{s.t.} & 0.6x_1 + 2.8x_2 \geq 25 \\ & 3x_1 + 7x_2 \leq 70 \\ & \sqrt{x_1} + \sqrt{x_2} \leq 4.05 \\ & x_1 + x_2 \leq 10 \\ & x_1, \; x_2 \geq 0 \\ & x_1, \; x_2 \in \mathbb{N} \end{array},$$

in which the constraint regarding to the available capital to invest is taken into account in its inequality version.

Among the possible ones, as initial feasible point we take $(0, 10)$. The constraint related to the transaction costs and the taxation are both satisfied; so, at first we let $x^* = (0, 10)$ and $f^* = 100$ (see step 1), and then we calculate the gradient in the same point, obtaining $(-10, 20)$ (see step 4). Finally, we add the related inequality

$$-10x_1 + 20x_2 < 200$$

to the auxiliary system in step 5.

The next feasible point we can consider among the ones solving the auxiliary system is $(0, 9)$; the costraints in step 2 and in step 3 are both satisfied. As $f(0, 9) = 81 < f^*$, we updated $x^* = (0, 9)$, $f^* = 81$, and we calculate again the gradient of the objective function in that point, obtaining $(-9, 18)$. Finally, we add the following corresponding (second) inequality

$$-9x_1 + 18x_2 < 162$$

to the auxiliary system in step 5.

The next solution of the updated auxiliary system we take into account (always among the existing ones) is $(2, 9)$. This point does not satisfy the constraint in step 2 (the one concerning with the taxation); so, now, we calculate the gradient of $f_2(\cdot)$ in this point, obtaining $(1, 1)$, and we add the third inequality

$$x_1 + x_2 < 11$$

to the auxiliary system in step 5.

The next point which solves the updated auxiliary system is $(1, 9)$; both the constarints in step 2 and in step 3 are satisfied. As $f(1, 0) = 72.6 < f^*$, we update $x^* = (1, 9)$ and $f^* = 72.6$; further, we calculate the gradient of the objective function in $(1, 9)$, obtaining $(-7.8, 17)$, and we add to the auxiliary system the new inequality

$$-7.8x_1 + 17x_2 < 145.2.$$

Now, the feasible set of the newly updated auxiliary system is empty; so, as $f^* \neq +\infty$, $x^* = (1, 9)$ is the optimal portfolio (see step 7), whose variance is equal to 72.8.

In Figure 1 we graphically represent the searching process of the optimal point (the dashed area indicates the feasible set determined without taking into account the integrity constraints).

In order to solve such kind of portfolio selection problems, a proper computer code has been developed,[7] in which CPLEX has been used for solving at each iteration the mixed-integer linear programming subproblem associated to the auxiliary system.

Notice that other efficient methods for solving optimization problems have been recently proposed (see [Andramonov *et al.*, 1999], [Andramonov, 2002a] and [Andramonov, 2002b]); in general, they need really weak assumptions regarding to the objective function and to the ones implied in the constraints.

5 Final remarks

As far is concerned to the mathematical programming problem presented in section **2** and to the related results proposed in section **3**, we give the following final remarks:

[7] It has been realized in the frame of the *Tacis ACE Programme - Action for cooperation in the field of economics "Financial Optimization in the New Independent State Financial Institutions"*.

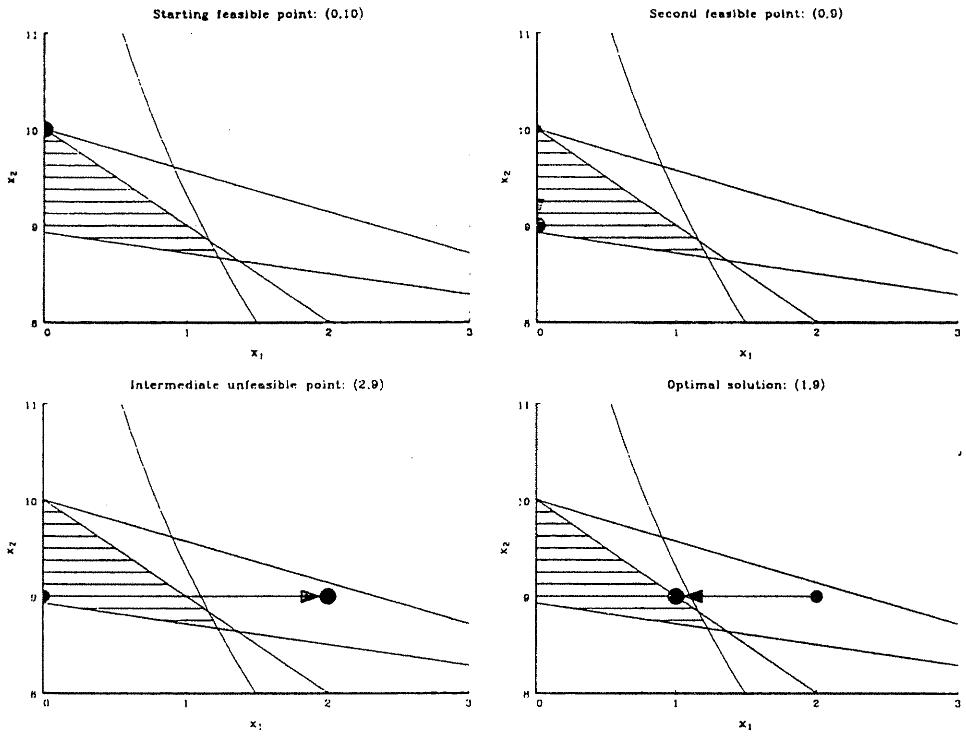

Fig. 1. Representation of the searching process of the optimal point.

- we introduce two non-linear constraints in order to take into account the presence of transaction costs and taxation; these constraints do not require any particular hypotheses, being sufficient to suppose the differentiability and the pseudo-convexity of the corresponding functions;
- for finding the optimal solution of the considered programming problem it is not necessary to elaborate specific methods, but it is enough to apply the combination of well known techniques: the branch and bound, the cutting plane ones (properly adjusting them to non-linear programming) and the sub-gradient one;
- the solving algorithm we propose does not require the objective function is quadratic, being sufficient to assume only its differentiability and pseudo-convexity;
- the method of portfolio selection we propose provides also some information about the "economic compatibility" of the values of the parameters π, α and β (assigned by the investor) with the real economy in which the investor acts; indeed, if the choice of these parameters is incoherent with the considered economic-financial

system, then the set of the feasible solutions could have poor financial sense or even could be empty.

With regard to the main directions of the future developments of the portfolio selection model and of the programming approach we presented in this paper, they can be summarized as follows:

- to generalize both the selection model and the solving algorithm in order to take into account an objective function and functions related to the transaction costs and the taxation which are not necessarily pseudo- or quasi-convex;
- to analyze the sensitivity of the optimal solution with respect to the integer number of assets constituting each lot; notice that these values are exogenous to the portfolio selection process, and that they could be an "interesting" instrument of financial policy for the stock exchange Authorities given their capability to influence the optimal portfolio selected by the investor;
- to make a suitably large number of applications of our approach to portfolio selection problems in presence of transaction costs and taxation in order to compare its results (and the related effectiveness) with the ones forecoming from the classical methods of selection of portfolio.

References

[Andramonov, 2002a] Andramonov, M.Y. (2002). *Methods of Minimization for some Classes of Generalized Convex Functions*. DAS Editors, Kazan (in Russian).

[Andramonov, 2002b] Andramonov, M.Y. (2002). A Survey of Methods of Abstract Convex Programming. *Journal of Statistics and Management Systems - Academic Forum*, to appear.

[Andramonov *et al.*, 1999] Andramonov, M.Y., Rubinov, A., and Glover, B. (1999). Cutting Angle Methods in Global Optimization. *Applied Mathematics Letters*, 12(3):95–100.

[Avella, 1990] Avella, P. (1990). Portfolio Selection: un Modello di Programmazione Mista-Intera risolto con un Algoritmo Branch and Bound. *Ricerca Operativa*, 56:3–28 (in Italian).

[Canestrelli *et al.*, 1992] Canestrelli, E., and Corazza, M. (1992). Un Modello Risolutivo a Variabili Miste-Intere per la Selezione di Portafoglio in Media-Varianza. *Atti del Sedicesimo Convegno A.M.A.S.E.S.*, Treviso, 203–218 (in Italian).

[Corazza, 1991] Corazza, M. (1991). Considerazioni sulla Applicazione di un Algoritmo di tipo "Branch and Bound" ad un Modello a Variabili Miste per la Selezione di Portafoglio in Media-Varianza. *Rendiconti del Comitato per gli Studi Economici*, XXIX:63–82 (in Italian).

[Corazza *et al.*, 1999] Corazza, M., and Favaretto, D. (1999). Approaching Mixed-Integer Nonlinear Mean-Variance Portfolio Selection. In Giorgi, G. and Rossi, F. (eds.) *Generalized Convexity and Optimization for Economic and Financial Decisions*, Pitagora Editrice, Bologna, 137–154.

[Demyanov *et al.*, 1985] Demyanov, V.F., and Vasiliev, L.V. (1985). *Nondifferentiable Optimization.* Springer-Verlag, Berlin.

[Elton *et al.*, 1984] Elton, E.J., and Gruber, M.J. (1984). *Modern Portfolio Theory and Investment Analysis.* John Wiley Sons, New York.

[Kelley, 1960] Kelley, J.E. (1960). The Cutting Plane Method for Solving Convex Programs. *Journal of the SIAM*, 8:703–712.

[Markowitz, 1989] Markowitz, H.M. (1989). *Mean-Variance Analysis in Portfolio Choice and Capital Markets.* Basil Blackwell, Oxford.

[Markowitz, 1991] Markowitz, H.M. (1991). *Portfolio Selection. Efficient Diversification of Investments.* Basil Blackwell, Oxford.

[Martello *et al.*, 1990] Martello, S., and Toth, P. (1990). *Knapsack Problems. Algorithms and Computer Implementations.* Wiley, New York.

[Nemhauser *et al.*, 1988] Nemhauser, G.L., and Wolsey, L.A. (1988). *Integer and Combinatorial Optimization.* Wiley, New York.

[Omprakash *et al.*, 1985] Omprakash, K.G., and Ravidan, A. (1985). Branch and Bound Experiments in Convex Nonlinear Integer Programming. *Management Science*, 31(12):1533–1546.

[Szegö, 1980] Szegö, G. P. (1980). *Portfolio Theory. With Application to Bank Asset Management.* Academic Press, New York.