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Properties of some generalized means for positive sequences

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Abstract. This paper proposes a simple procedure, in order to aggregate a finite number of real nonnegative values into a unique indicator. Possible applications of this tool can be found in social sciences, including demography, sociology, etc. In particular, the indicator may represent a generalized mean, to be used as an aggregate measure combining several inhomogeneous parameters. Observe that some alternatives to this approach, including complex multicriteria or multiobjective methods, are often discarded by stakeholders (say politicians, public administrators, managers, etc.), since the latters are typically keen on taking decisions based on a reduced number of parameters (possibly only one). Hence, administrators often show some reluctance to adopt those methods which provide multiple alternatives, e.g. a Pareto front, since this fact implies an additional process of selection.

Keywords: Aggregation of indicators, Ranking among measures, Mean weighted values, Concave problems.

JEL Classification Numbers: C44, C65.

1 Introduction

In this paper we analyze and test properties of a generalized procedure, which is used to aggregate positive real numbers. Our method is meant in order to aggregate a finite number of real parameters (hereafter the *Elementary Indicators* - EI) into a unique nonnegative indicator. The technique we adopted for aggregation is intended to provide a reliable and simple tool (see also [1]). This tool can be embedded within a decision process, where stakeholders are often eager to base their preferences on simple and reliable decision support systems.

In Section 2 we give the main motivations for our proposal, while Section 3 details some theoretical results. Section 4 contains further theoretical analysis relative to the results in Section 3. Finally, Section 5 provides an extension of our proposal.

2 Motivations

Let us consider a finite set of $m \geq 1$ values $w_1, \dots, w_m$, and a finite set of m EI. Let the entries of the vector $x \in \mathbb{R}^m$ correspond to the EI. We also define the quantity $|x|_p$ as in

$$|x|_p \doteq \left[\sum_{i=1}^{m+1} w_i x_i^p \right]^{1/p}, \qquad p \in \mathbb{R}, \tag{2.1}$$

being $x_{m+1} = \varepsilon$, with $\varepsilon \in (0,1)$. Moreover, given $p_1 \in \mathbb{R}$, we assume that the vector x, the vector $w = (w_1, \dots, w_m)^T$ and the constant value w_{m+1} satisfy

$$\begin{cases} w \in \mathbb{R}^m \setminus \{0\} : \quad w_{m+1} = \varepsilon^{|p_1|}, \quad \sum_{i=1}^{m} w_i = 1 - \varepsilon^{|p_1|}, \quad w_i \geq 0, \quad 0 \leq x_i \leq 1, \quad i = 1, \dots, m \\ x_i = 0 \quad \Longrightarrow \quad w_i = 0, \qquad i \in \{1, \dots, m\}. \end{cases} \tag{2.2}$$

Observe that when $w \geq 0$ and $p \geq 1$ then (2.1) defines a generalized weighted p-mean of the vector $(x^T, \varepsilon)^T$. Conversely, in case $w \geq 0$ and $p \in (0,1)$, the triangular inequality might not be satisfied by the function $|\cdot|_p$ in (2.1), meaning that $|\cdot|_p$ does not represent a *norm*. Similarly, note that also in case $p \leq 0$ in (2.1), then $|x|_p$ no more represents a norm, because the implication

$$|x|_p = 0 \quad \Longleftrightarrow \quad x = 0$$

is no more guaranteed to hold. Nevertheless, there might be reasons which possibly advise setting $p < 0$ in our analysis, renouncing to work with a norm. Indeed, on one hand the use of negative values for p allows specific achievements, which generalize standard results yielded by the use of norms. On the other hand, the expression in (2.1) generalizes the well known p-mean (Hölder p-mean)

$$q_p(x) \doteq \left[\frac{1}{m+1} \sum_{i=1}^{m+1} x_i^p \right]^{1/p}, \qquad p \in \mathbb{R}, \tag{2.3}$$

since (2.1) allows possibly $w_i \neq 1/(m+1)$. Interestingly enough, there are very special values for p in (2.3) such that $q_p(x)$ represents a versatile tool for many practical applications. As an example, assume for simplicity $x > 0$; then, the next table shows the role played by $q_p(x)$ in case of some relevant values for p:

Value of p		**Role played by** $q_p(x)$
$p = -1$	$\Longrightarrow$	armonic mean of x entries
$p \to 0$	$\Longrightarrow$	geometric mean of x entries
$p = +1$	$\Longrightarrow$	arithmetic mean of x entries
$p = +2$	$\Longrightarrow$	mean of squares of x entries
$p \to -\infty$	$\Longrightarrow$	$\min_i\{x_i\}$ (i.e. the minimum of x entries)
$p \to +\infty$	$\Longrightarrow$	$\max_i\{x_i\}$ (i.e. the maximum of x entries)

Hence, the role played by the values $p = -1$ and $p = +1$ has frequently a terrific impact on several applications. This motivates the specific additional investigation in the current paper, where intervals for the parameter p nearby -1 and $+1$ are analyzed, in the light of possibly preserving an ordering among different generalized means.

As an example, in case $p_1, p_2 > 0$ the fulfillment of the ordering $|x|_{p_1} \geq |x|_{p_2}$ in general depends both on the choice of vectors x and w, as well as on the parameter ε. On the other hand, we want to show that properly setting negative and positive values for p_1 and p_2, then the fulfillment of the inequality $|x|_{p_1} \geq |x|_{p_2}$ can be easily controlled.

We also warn the reader about the fact that the value of ε in (2.2) is introduced to guarantee the fulfillment of the hypotheses in the next Proposition 3.1. Thus, ε plays a merely technical role, so that x_{m+1} and w_{m+1} in (2.1) and (2.2) may be interpreted as fictitious $(m+1)$-th entries of vectors x and w, respectively. For further properties of $|x|_p$ in (2.1) the reader may also refer to [2], pages 117–127.

3 Some theoretical results

As a preliminary fact, selecting any nonzero values p_1 and p_2 for the parameter p in (2.1), the next relation immediately holds

$$|x|_{p_1} \geq |x|_{p_2} \quad \Longleftrightarrow \quad \frac{1}{p_1} \ln \left[\sum_{i=1}^{m+1} w_i x_i^{p_1} \right] \geq \frac{1}{p_2} \ln \left[\sum_{i=1}^{m+1} w_i x_i^{p_2} \right]. \tag{3.1}$$

Let us now consider the next proposition, which provides some hints for the choice of the parameter p in (2.1). In particular, we want to infer some rules, which possibly suggest values for p_1 and p_2 such that relation (3.1) holds, *regardless of the choice of values* $w_1, \ldots, w_m, w_{m+1}$ satisfying relations (2.2). This might be of somewhat importance in case we desire to aggregate the entries of vector x representing the EI, by using different values of p in (2.1), say p_1 and p_2, with the aim of maintaining a predictable ordering between the quantities $|x|_{p_1}$ and $|x|_{p_2}$.

Indeed, the quantities $w_1, \ldots, w_m$ in our application represent weights. These weights are often the result of practical arrangements, or they often depend on a series of compromises, so that they are subject to changes, according with possible different scenarios. As a result, we can be interested about aggregating in (3.1) the values of the entries of x (i.e. the EI), through the weights $w_1, \ldots, w_m$, so that respectively setting $p = p_1$ and $p = p_2$ in (2.1) we can predict the fulfillment of inequality $|x|_{p_1} \geq |x|_{p_2}$, regardless of the choice of $w_1, \ldots, w_m$.

Proposition 3.1 *Suppose the real values $w_1, \ldots, w_m, w_{m+1}$ satisfy relations* (2.2). *Assume that*

1. $p_2 \leq -1 < p_1 < 0$,

2. $|p_2| \leq |p_1| \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right] / \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right]$,

3. $\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} > 1$.

Then, we have

$$|x|_{p_1} \geq |x|_{p_2}. \tag{3.2}$$

Proof: Observe that since $x_{m+1} = \varepsilon$, $w_{m+1} = \varepsilon^{|p_1|}$ and $(w_1, \ldots, w_m) \neq 0$, then (2.2) trivially implies *3.* Furthermore, since p_1 and p_2 are negative values, then relations (3.1) (i.e. equivalently relation (3.2)) may be rewritten as

$$\frac{1}{|p_1|} \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right] \leq \frac{1}{|p_2|} \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_2|}} \right]. \tag{3.3}$$

Since now $|p_2| \geq 1 > |p_1| > 0$, recalling that $0 \leq x_i \leq 1$ for any $1 \leq i \leq m+1$, we obtain for any i satisfying $w_i \neq 0$

$$\frac{w_i}{x_i^{|p_2|}} \geq \frac{w_i}{x_i} \geq \frac{w_i}{x_i^{|p_1|}}. \tag{3.4}$$

Thus, using inequalities (3.4), condition (3.3) is surely satisfied provided that there exist values $w_1, \ldots, w_m, w_{m+1}$ fulfilling

$$\frac{1}{|p_1|} \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right] \leq \frac{1}{|p_2|} \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right] \leq \frac{1}{|p_2|} \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_2|}} \right]. \tag{3.5}$$

Now, the rightmost inequality in (3.5) directly follows from (3.4), while from *3.* the leftmost inequality in (3.5) requires

$$|p_2| \leq |p_1| \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right] / \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right], \tag{3.6}$$

which is indeed fulfilled by *1.* and *2.* □

Lemma 3.2 *Assume the real values $w_1, \ldots, w_m, w_{m+1}$ satisfy conditions* (2.2)*. Let be given the real parameters ε, p_1 and p_2, where $p_2 \leq -1 < p_1 < 0$, and $\varepsilon \in (0,1)$. Then, there exist nonempty sets $A(p_1, \varepsilon) \subset \mathbb{R}^m$ and $B(p_1, \varepsilon) \subset \mathbb{R}$, depending on p_1 and ε, such that for any $w \in A(p_1, \varepsilon)$ and for any $p_2 \in B(p_1, \varepsilon)$, the hypotheses 1., 2., 3. of Proposition 3.1 are fulfilled.*

Proof: Condition *1.* of Proposition 3.1 trivially holds. Moreover, by (2.2) the choice of x_{m+1} and w_{m+1}, along with relation $(w_1, \ldots, w_m) \neq 0$, straightforwardly guarantee that the condition *3.* in Proposition 3.1 is satisfied, too. In addition, again using conditions (2.2), and recalling that for a concave function $f : \mathbb{R} \to \mathbb{R}$ the next *Jensen inequality* holds

$$f\left(\sum_{i=1}^{m+1} \beta_i z_i\right) \geq \sum_{i=1}^{m+1} \beta_i f(z_i), \qquad \sum_{i=1}^{m+1} \beta_i = 1, \quad 0 \leq \beta_i \leq 1, \quad i = 1, \ldots, m+1,$$

we have the following relations (where $f(z) = z^{|p_1|}$)

$$\left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right)^{|p_1|} \geq \sum_{i=1, w_i \neq 0}^{m+1} w_i \left(\frac{1}{x_i}\right)^{|p_1|} = \sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \tag{3.7}$$

so that by *3.* of Proposition 3.1

$$\ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right)^{|p_1|} \geq \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right) > 0.$$

This equivalently implies that the condition

$$|p_1| \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right) / \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right) > 1$$

is fulfilled. Moreover, this also implies that there are negative values of p_2 satisfying

$$1 \leq |p_2| < |p_1| \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right) / \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right),$$

i.e. there are values of p_2 such that the condition *2.* in Proposition 3.1 holds. As a consequence, since $(w_1, \dots, w_m) \neq 0$ then the sets

$$A(p_1, \varepsilon) \doteq \left\{ w \in \mathbb{R}^m : \sum_{i=1}^{m} w_i = 1 - \varepsilon^{|p_1|}, \quad w_i \geq 0, \quad i = 1, \dots, m, \quad \varepsilon \in (0,1), \ p_1 \in (-1, 0) \right\} \tag{3.8}$$

$$B(p_1, \varepsilon) \doteq \left\{ p \in \mathbb{R} : \ p < 0, \quad 1 \leq |p| \leq |p_1| \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right] / \ln \left[\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_2|}} \right], \quad \forall w \in A(p_1, \varepsilon) \right\} \tag{3.9}$$

are nonempty, and for any $w \in A(p_1, \varepsilon)$ and $p_2 \in B(p_1, \varepsilon)$ the hypotheses of Proposition 3.1 are satisfied. □

Given the parameters p_1 and ε, Figure 3.1 plots an example (shaded area) for the set $A(p_1, \varepsilon)$.

Corollary 3.3 *Under the hypotheses of Lemma 3.2, for any choice of the parameters p_1 and ε the sets $A(p_1, \varepsilon)$ and $B(p_1, \varepsilon)$ defined in* (3.8) *and* (3.9) *are compact and convex, being $A(p_1, \varepsilon)$ also a polyhedron.*

Proof: The result for $A(p_1, \varepsilon)$ simply follows by observing that $A(p_1, \varepsilon)$ is equivalently described by a finite set of linear equalities and inequalities. Moreover, $B(p_1, \varepsilon)$ is defined by a set of linear inequalities whose cardinality is possibly not finite. Finally, the compactness of $A(p_1, \varepsilon)$ and $B(p_1, \varepsilon)$ follows directly from (3.8) and (3.9).

□

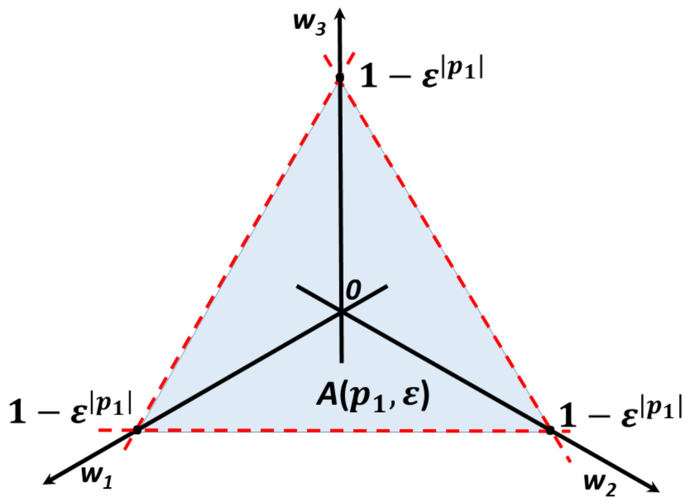

Figure 3.1: An example of the set $A(p_1, \varepsilon)$ in (3.8) (shaded area), in case $m = 3$: this set is clearly a polyhedron in $\mathbb{R}^3$.

As we are going to comment later on, the results in Corollary 3.3 have the importance to guarantee a possibly simpler solution for some optimization problems we formulate in the sequel. In particular, according with Lemma 3.2, in order to estimate an interval of values for the negative parameter p_2 such that

$$1 \leq |p_2| < |p_1| \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right) / \ln \left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right),$$

we may start from considering the sets $A(p_1, \varepsilon)$ and $B(p_1, \varepsilon)$. Then, let us define the function $\varphi_{p_1}(\varepsilon)$ of ε, with $\varphi_{p_1} : \mathbb{R} \to \mathbb{R}$, which solves for a given value of ε and p_1 the optimization problem

$$\begin{aligned} \varphi_{p_1}(\varepsilon) \;\in\; \arg\min_{w_1,\ldots,w_m} \quad & \left\{ |p_1| \ln\left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i} \right) / \ln\left(\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right) \right\} \\ & \sum_{i=1, w_i \neq 0}^{m} w_i = 1 - \varepsilon^{|p_1|} \\ & w_i \geq 0, \quad i = 1, \ldots, m. \end{aligned} \tag{3.10}$$

We recall that in (3.10) the parameters ε and p_1 respectively range in $(0, 1)$ and $(-1, 0)$. Observe that for given ε and p_1, the problem (3.10) always admits solutions, since its objective function is continuous and the set $A(p_1, \varepsilon)$ in (3.8) is compact by Corollary 3.3.

As an example, in Figure 3.2 we have set $m = 3$, with $x_1 = 1/2$, $x_2 = 2/3$, $x_3 = 3/4$ and $p_1 \in \{-0.1, -0.3, -0.5, -0.7\}$; then, we have plot the function $\varphi_{p_1}(\varepsilon)$ versus ε. For the solution of the (possibly) nonconvex constrained optimization problem (3.10) we used Matlab [3] built-in function *fmincon()*, adopting the default option of *interior-point-methods* [4] and standard Matlab settings. We remark that the objective function in (3.10) is possibly nonconvex, while the feasible set is a (convex) polyhedron. Thus, the interior-point algorithm adopted in *fmincon()* is well–posed, being the interior of the set $A(p_1, \varepsilon)$ in (3.8) nonempty (see Lemma 3.2). Furthermore, *fmincon()* uses an exact optimization method, but for given p_1 and ε it possibly provides a local minimum in (3.10), which might not be also a global minimum. Thus, Figure 3.2 simply reports a numerical experience to validate the conclusions of Lemma 3.2, where for our purposes the detection of local minima might be a satisfactory achievement.

Similarly, Figure 3.3 reports a numerical experience on the guidelines of Figure 3.2; however, Figure 3.3 makes reference to a larger instance where $m = 20$ and $x_i = i/(i + 1)$, $i = 1, \ldots, m$.

We highlight that the investigation in this section may have a dramatic impact in practice. Indeed, Proposition 3.1 allows to tune both the choice of the weights $\{w_i\}$ and the parameter p on the problem in hand, in order to provide a fruitful tool to aggregate EI, in the light of supporting decisions for politicians and other stakeholders. Furthermore, results of Proposition 3.1 confirm that the fulfillment of condition (3.2) can be ensured under *any choice* of coefficients $\{w_i\}$ that satisfy (2.2). This means that given p_1, the process of assessing p_2 can, to some extent, be considered robust, with respect to changes of $\{w_i\}$. Moreover, from Figures 3.2 and 3.3 it is evident that a relatively small value for the parameter ε is advisable, since this allows a larger range of feasible values for p_2 satisfying *2.* of Proposition 3.1.

4 Issues on the solution of problem (3.10)

This section is specifically devoted to analyze the solution of the nonconvex constrained optimization problem (3.10), for given p_1 and ε. In particular, we prove that the objective function in (3.10) is strictly concave on the feasible set, so that possible solutions of (3.10) are located on vertices of its feasible polyhedron. In this regard, we have the following result.

Lemma 4.1 *Given $p_1 \in (-1, 0)$ and $\varepsilon \in (0, 1)$, let the assumptions of Proposition 3.1 hold, and assume $A(p_1, \varepsilon)$ is the polyhedron defined in* (3.8)*. Let $w \in \mathbb{R}^m$ such that*

$$\sum_{i=1, w_i \neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} = \gamma, \qquad \gamma > 1. \tag{4.1}$$

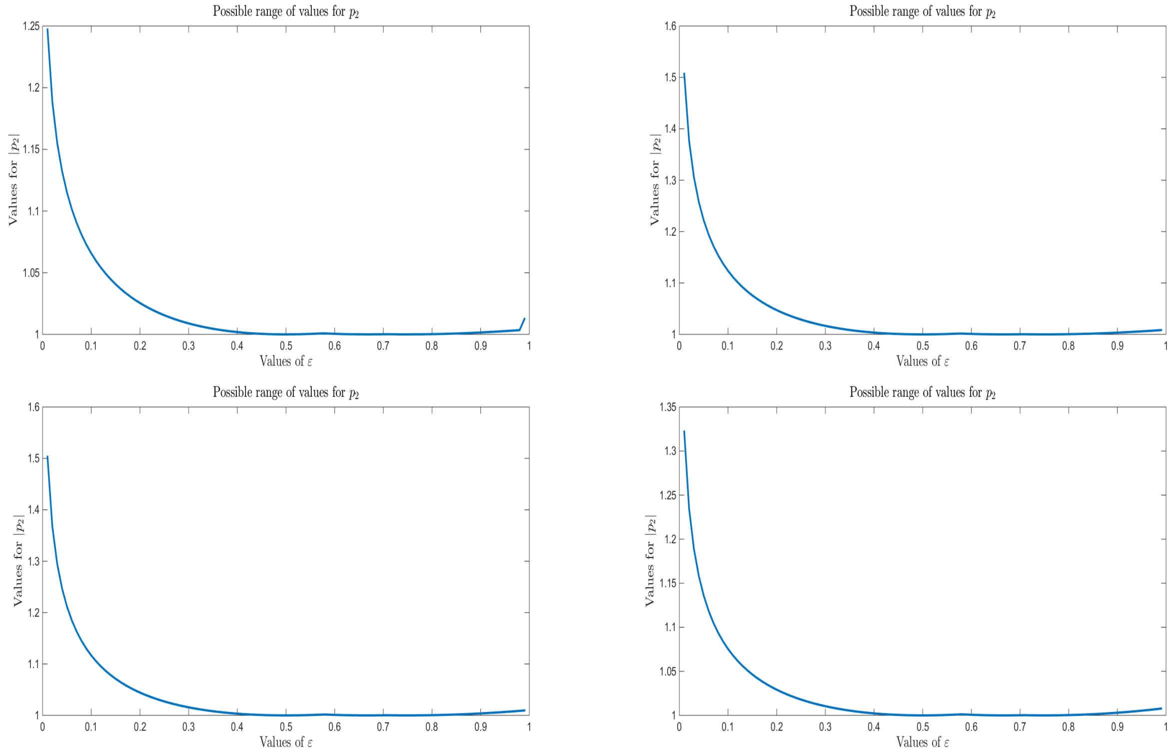

Figure 3.2: Plot of the function $\varphi_{p_1}(\varepsilon)$ in (3.10), in case $m = 3$ and $p_1 = -0.1$ (*top left*), $p_1 = -0.3$ (*top right*), $p_1 = -0.5$ (*bottom left*), $p_1 = -0.7$ (*bottom right*), when $\varepsilon \in (0, 1)$.

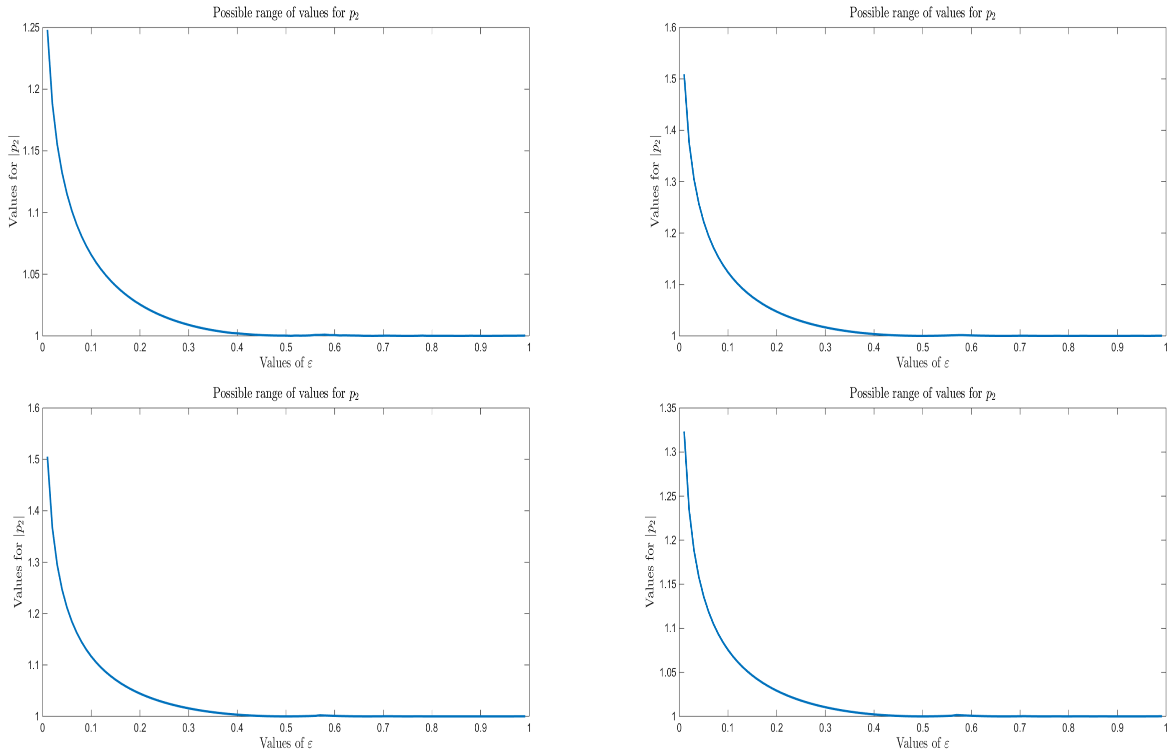

Figure 3.3: Plot of the function $\varphi_{p_1}(\varepsilon)$ in (3.10), in case $m = 20$ and $p_1 = -0.1$ (*top left*), $p_1 = -0.3$ (*top right*), $p_1 = -0.5$ (*bottom left*), $p_1 = -0.7$ (*bottom right*), when $\varepsilon \in (0, 1)$.

Then, the objective function in (3.10) *is strictly concave on the compact convex set (polyhedron)*

$$\mathcal{A}(p_1,\varepsilon) \ \doteq\ \left\{ w \in \mathrm{I\!R}^m \ : \ w \in A(p_1,\varepsilon), \quad \sum_{i=1, w_i\neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} = \gamma \right\}.$$

Proof: First observe that for any $w \in A(p_1,\varepsilon)$ the objective function in (3.10) can be written as

$$|p_1| \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i} \right] / \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right] \ = \ |p_1| \log_{\left[\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}} \right]} \left[\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i} \right]. \qquad (4.2)$$

Then, recalling the definition of concavity for the function $g : \mathrm{I\!R} \to \mathrm{I\!R}$, on the convex set Ω, i.e.

$$g\left[\beta z^{(1)} + (1-\beta) z^{(2)}\right] \geq \beta g\left[z^{(1)}\right] + (1-\beta) g\left[z^{(2)}\right], \qquad \forall \beta \in [0,1], \qquad \forall z^{(1)}, z^{(2)} \in \Omega,$$

and defining, for any $w \in A(p_1,\varepsilon)$, the two linear functions

$$\begin{cases} b(w) \doteq \displaystyle\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i} \\ c(w) \doteq \displaystyle\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}}, \end{cases}$$

by *3.* of Proposition 3.1 we have $b(w) \geq c(w) > 1$. In addition, by (4.1) we have

$$c(w) = \gamma, \qquad \forall w \in \mathcal{A}(p_1,\varepsilon),$$

so that for any $w^{(1)}, w^{(2)} \in \mathcal{A}(p_1,\varepsilon)$, $0 < \beta < 1$, we obtain

$$\beta c\left[w^{(1)}\right] + (1-\beta) c\left[w^{(2)}\right] = c\left[w^{(1)}\right] = c\left[w^{(2)}\right] > 1.$$

Now, being $b(w)$ and $c(w)$ linear functions with respect to w, for any $w \in \mathcal{A}(p_1,\varepsilon)$ relation (4.1) and the strict concavity of the logarithm yield (see (4.2))

$$\begin{aligned}
& |p_1| \log_{c\left[\beta w^{(1)} + (1-\beta) w^{(2)}\right]} \left\{ b\left[\beta w^{(1)} + (1-\beta) w^{(2)}\right] \right\} \ = \\
= \ & |p_1| \log_{\left\{\beta c\left[w^{(1)}\right] + (1-\beta) c\left[w^{(2)}\right]\right\}} \left\{ \beta b\left[w^{(1)}\right] + (1-\beta) b\left[w^{(2)}\right] \right\} \\
> \ & |p_1| \left\{ \beta \log_{\left\{\beta c\left[w^{(1)}\right] + (1-\beta) c\left[w^{(2)}\right]\right\}} \left\{ b\left[w^{(1)}\right] \right\} + (1-\beta) \log_{\left\{\beta c\left[w^{(1)}\right] + (1-\beta) c\left[w^{(2)}\right]\right\}} \left\{ b\left[w^{(2)}\right] \right\} \right\} \\
= \ & |p_1| \left\{ \beta \log_{c\left[w^{(1)}\right]} \left\{ b\left[w^{(1)}\right] \right\} + (1-\beta) \log_{c\left[w^{(2)}\right]} \left\{ b\left[w^{(2)}\right] \right\} \right\}.
\end{aligned}$$

Finally, this proves that the objective function in problem (3.10) is strictly concave over the polyhedron $\mathcal{A}(p_1,\varepsilon)$, whose compactness is yielded by the compactness of the set $A(p_1,\varepsilon)$. □

4.1 On some properties of problem (3.10)

In this section we focus on the vertices of the feasible polyhedron $\mathcal{P}$ in problem (3.10) (i.e. the set $A(p_1, \varepsilon)$ in (3.8)), as well as the vertices of the polyhedron $\mathcal{A}(p_1, \varepsilon)$ in Lemma 4.1. On this purpose, we respectively indicate by (I), (II) and (III) the next three sets of constraints:

$$\begin{cases} (I) & w_i \geq 0, \qquad i = 1, \ldots, m, \\ (II) & \displaystyle\sum_{i=1}^{m} w_i = 1 - \varepsilon^{|p_1|}, \\ (III) & \displaystyle\sum_{i=1, w_i \neq 0}^{m} \frac{w_i}{x_i^{|p_1|}} = \gamma, \qquad \gamma > 1. \end{cases}$$

$\mathcal{P}$ is defined as the intersection of the polyhedron (I) with the hyperplane (II), and in order to give a geometric description of $\mathcal{P}$ it suffices to recur to Figure 3.1. Conversely, $\mathcal{A}(p_1, \varepsilon)$ is obtained by the intersection of $\mathcal{P}$ with the hyperplane (III). Observe that the normal vector to the hyperplanes *(II)* and *(III)* is respectively given by (assume without loss of generality that $w_i \neq 0$, $1 \leq i \leq m$)

$$u_{(II)} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^m, \qquad u_{(III)} = \begin{pmatrix} \frac{1}{x_1^{|p_1|}} \\ \vdots \\ \frac{1}{x_m^{|p_1|}} \end{pmatrix} \in \mathbb{R}^m.$$

Thus, to better investigate the geometry of the sets $A(p_1, \varepsilon)$ and $\mathcal{A}(p_1, \varepsilon)$ we distinguish between two cases:

- (II) and (III) are parallel (possibly coincident): this happens if and only if $u_{(II)}$ and $u_{(III)}$ are parallel, i.e. if and only if $x_1 = \cdots = x_m$. In the latter case, if $x_1^{|p_1|}\gamma = \cdots = x_m^{|p_1|}\gamma = 1 - \varepsilon^{|p_1|}$, then the polyhedron $\mathcal{A}(p_1, \varepsilon)$ has the same vertices of the polyhedron $\mathcal{P}$ in Figure 3.1, while if $x_1^{|p_1|}\gamma = \cdots = x_m^{|p_1|}\gamma \neq 1 - \varepsilon^{|p_1|}$ then the polyhedron $\mathcal{A}(p_1, \varepsilon)$ is empty;
- $u_{(II)}$ and $u_{(III)}$ are not parallel; then for suitable values of the parameter γ the polyhedron $\mathcal{P}$ has a nonempty intersection with the hyperplane (III). To determine in this case the vertices of $\mathcal{A}(p_1, \varepsilon)$, let us first consider the vertices of $\mathcal{P}$, i.e. the points

  $$v_i = (1 - \varepsilon^{|p_1|})e_i, \qquad i = 1, \ldots, m, \quad w_i \neq 0,$$

  being e_i the i-th real unit vector. Note that the hyperplane (III) includes the point v_i provided that

  $$\gamma = \frac{1 - \varepsilon^{|p_1|}}{x_i^{|p_1|}} > 1. \tag{4.3}$$

  Thus, if $u_{(II)}$ and $u_{(III)}$ are not parallel, in order the polyhedron $\mathcal{A}(p_1, \varepsilon)$ to be nonempty, the parameter k must satisfy the inequalities*

  $$\min_{i=1,\ldots,m, w_i \neq 0} \left\{ \frac{1 - \varepsilon^{|p_1|}}{x_i^{|p_1|}} \right\} \leq \gamma \leq \max_{i=1,\ldots,m, w_i \neq 0} \left\{ \frac{1 - \varepsilon^{|p_1|}}{x_i^{|p_1|}} \right\}. \tag{4.4}$$

*We highlight that by (4.3) the leftmost inequality in (4.4) also fulfils relation *3.* of Proposition 3.1.

In addition, from Figure 3.1 we can immediately infer that the vertices (if any) of $\mathcal{A}(p_1,\varepsilon)$ can lie only on the boundary (dashed lines of Figure 3.1) of the polyhedron $\mathcal{P}$. As a consequence, since by Lemma 4.1 the objective function in (3.10) is strictly concave on $\mathcal{P}$ (inasmuchas as $\mathcal{A}(p_1,\varepsilon)$ is a polyhedron and $\mathcal{A}(p_1,\varepsilon) \subseteq \mathcal{P} = A(p_1,\varepsilon)$), and recalling that when γ ranges in the interval (4.4) then any point on the boundary of $\mathcal{P}$ can be a vertex of $\mathcal{A}(p_1,\varepsilon)$, we have the following conclusion.

Proposition 4.2 *Consider the concave optimization problem* (3.10)*; then, given the parameters* $p_1 \in (-1,0)$ *and* $\varepsilon \in (0,1)$

- *the global minima of the objective function are among the vertices* $\{v_i\}$ *of* $A(p_1,\varepsilon)$*, with*

$$v_i = \left(1-\varepsilon^{|p_1|}\right) e_i, \qquad i=1,\dots,m,\ w_i \neq 0;$$

- *the objective function in* (3.10) *satisfies*

$$\lim_{\varepsilon\to 0^+} |p_1| \ln\left(\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i}\right) / \ln\left(\sum_{i=1,w_i\neq 0}^{m+1} \frac{w_i}{x_i^{|p_1|}}\right) = +\infty.$$

Proof: By Lemma 4.1 problem (3.10) is strictly concave and its feasible polyhedron is compact, which implies that global solutions both exist and are located in one of the vertices $\{v_i\}$ (see also [5]).

As regards the second item, given the parameter $p_1 \in (-1,0)$, let $\hat{\imath} \in \{1,\dots,m\}$ be one of the indices such that $v_{\hat{\imath}}$ is a global minimum. Then, the value of the objective function in (3.10) at $v_{\hat{\imath}}$ is

$$|p_1| \frac{\ln\left[\dfrac{\varepsilon^{|p_1|}}{\varepsilon} + \dfrac{w_{\hat{\imath}}}{x_{\hat{\imath}}}\right]}{\ln\left[1 + \dfrac{w_{\hat{\imath}}}{x_{\hat{\imath}}^{|p_1|}}\right]}.$$

Thus, taking the limit $\varepsilon \to 0^+$ we obtain the result. □

Observe that the property at the second item of the last proposition has also a numerical evidence, considering small values of ε in the plots of Figures 3.2-3.3.

5 A further extension

We complete this paper by proving similar results with respect to Proposition 3.1, while setting positive values for the parameters p_1 and p_2. In this regard, again we preliminarily set $x_{m+1} = \varepsilon$, with $\varepsilon \in (0,1)$. Moreover, given $p_1 \in \mathbb{R}$ we assume that now (similarly to (2.2)) the vector $w \in \mathbb{R}^m$ and the constant value w_{m+1} satisfy relations

$$\begin{cases} w \in \mathbb{R}^m \setminus \{0\}: \ w_{m+1} = \frac{1}{\varepsilon^{|p_1|}}, \quad \displaystyle\sum_{i=1}^m w_i = 1 - \frac{1}{\varepsilon^{|p_1|}}, \quad w_i \geq 0, \quad 0 \leq x_i \leq 1,\ i=1,\dots,m, \\ x_i = 0 \quad \Longrightarrow \quad w_i = 0, \qquad i \in \{1,\dots,m\}. \end{cases} \tag{5.1}$$

Proposition 5.1 *Suppose the real values* $w_1,\dots,w_m,w_{m+1}$ *satisfy relations* (5.1)*. Assume that*

1. $0 < p_1 < 1 \leq p_2$,
2. $p_2 \leq p_1 \ln\left[\displaystyle\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i\right] / \ln\left[\displaystyle\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1}\right]$,

3. $\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} > 1.$

Then, we have

$$|x|_{p_1} \geq |x|_{p_2}. \tag{5.2}$$

Proof: The proof basically follows guidelines similar to those of Proposition 3.1. In particular, note that (5.1) implies $(w_1, \ldots, w_m) \neq 0$ so that *3.* holds. Furthermore, by (2.1) relation (5.2) is fulfilled if and only if the condition

$$\frac{1}{p_1} \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} \right] \geq \frac{1}{p_2} \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_2} \right] \tag{5.3}$$

holds. Observe that now by (5.1) and *1.* it is for any $i = 1, \ldots, m$

$$w_i x_i^{p_1} \geq w_i x_i \geq w_i x_i^{p_2}. \tag{5.4}$$

Moreover, we can conclude that (5.3) is surely satisfied in case there exist values $w_1, \ldots, w_m, w_{m+1}$ which yield

$$\frac{1}{p_1} \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} \right] \geq \frac{1}{p_2} \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i \right] \geq \frac{1}{p_2} \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_2} \right]. \tag{5.5}$$

Finally, the rightmost inequality in (5.5) directly follows from (5.1) and (5.4), while the leftmost inequality in (5.5) is a consequence of *2.* and *3.* □

Lemma 5.2 *Assume the real values* $w_1, \ldots, w_m, w_{m+1}$ *satisfy conditions* (5.1)*. Let be given the real parameters* ε, p_1 *and* p_2*, where* $0 < p_1 < 1 \leq p_2$*, and* $\varepsilon \in (0, 1)$*. Then, there exist nonempty sets* $\bar{A}(p_1, \varepsilon) \subset \mathbb{R}^m$ *and* $\bar{B}(p_1, \varepsilon) \subset \mathbb{R}$*, depending on* p_1 *and* ε*, such that for any* $w \in \bar{A}(p_1, \varepsilon)$ *and for any* $p_2 \in \bar{B}(p_1, \varepsilon)$*, the hypotheses 1., 2., 3. of Proposition 5.1 are fulfilled.*

Proof: The proof is similar to the one of Lemma 3.2, so that the hypotheses easily yield *1.* and *3.* of Proposition 5.1. As regards *2.* of Proposition 5.1, from the *Jensen inequality* for the concave function $f(z) = z^{p_1}$, $z \geq 0$, we have

$$\left(\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i \right)^{p_1} \geq \sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1},$$

so that by *3.* of Proposition 5.1

$$\ln \left(\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i \right)^{p_1} \geq \ln \left(\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} \right) > 0,$$

i.e.

$$p_1 \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i \right] / \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} \right] > 1.$$

Finally, the last inequality implies that there exist values for p_2 such that

$$1 \leq p_2 < p_1 \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i \right] / \ln \left[\sum_{i=1,w_i\neq 0}^{m+1} w_i x_i^{p_1} \right],$$

proving that the condition *2.* in Proposition 5.1 holds. As a consequence, the existence of the sets $\bar{A}(p_1, \varepsilon)$ and $\bar{B}(p_1, \varepsilon)$ is guaranteed. □

Corollary 5.3 *Under the hypotheses of Lemma 5.2, for any choice of the parameters p_1 and ε the sets $\bar{A}(p_1, \varepsilon)$ and $\bar{B}(p_1, \varepsilon)$ defined in Lemma 5.2 are compact and convex, being $\bar{A}(p_1, \varepsilon)$ also a polyhedron.*

Proof: The proof is almost identical to that of Corollary 3.3. □

6 Conclusions and future work

In this paper we analyzed a method to aggregate a finite number of real parameters into a unique nonnegative indicator. The analysis is carried on following a theoretical point of view. Nevertheless, the proposed technique can be easily embedded within different real decision processes, where stakeholders are often eager to base their preferences on simple and reliable decision support systems, rather than using a (possibly large) number of indicators.

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