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A unified framework for performance and risk attribution

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A unified frame work for performance and risk attribution

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Abstract
Investment performance evaluation is one of the pillars of finance and its techniques have refined throughout the years. This work focuses on the evaluation of the investment performance achieved through a top-down investment strategy analyzed using the Brinson model: a set of techniques that permits to algebraically examine the performance contributions of the investment decisions taken. The model, that originated in 1985, has been constantly refined throughout the years to overcome some of its major problems. In particular, this work analyzes the improvements that permit to apply the Brinson model to a multi-period timeframe and to a risk analysis process. Lastly, this work will present a new approach that adapts the Brinson model to a multi-period timeframe. This new approach refines some of the tools presented in the literature and analyzes the investment decisions from a risk-return perspective rather than a return-only perspective.

Keywords
Performance attribution, Risk attribution, Brinson Model.

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1 Introduction

Performance attribution is a discipline that allocates the active return of an investment among the investment decisions taken during the period. This subject consists of many tools and, in this article, we will focus on the Brinson model: a well-known approach for the algebraic decomposition of the active return attained through top-down investment decisions.

The article will encompass: the story of the Brinson model (briefly explained to keep track of the refinements underwent by this paradigm), the improvements proposed to some of its tools (increase in preciseness leads to a more correct evaluation) and, in conclusion, this article will present a new model for performance and risk attribution.

2 History of the model and its refinements

This paragraph is aimed to give a clear presentation of the Brinson model and the modifications it has underwent during the last 25 years.

In 1985 Gary Brinson and Nimrod Fachler issued a method to decompose the active return attained through top-down investment decisions[1]: the Brinson model. The idea behind it remains compelling: algebraically decompose the arithmetic active return of an investment into allocation and selection decisions[2]. In other words, the arithmetic active return is calculated as an arithmetical difference between the portfolio and the benchmark return: $AR_t = R_t - \overline{R}_t$.Then, allocation and selection effects (the decisions that an investor takes when following a top-down investment strategy) are consequently expressed as[3]:

$$Allocation \quad : \quad \sum_i (w_i - \overline{w}_i)(\overline{r}_i - \overline{R}_i) \tag{1}$$

$$Selection \quad : \quad \sum_i w_i(r_i - \overline{r}_i) \tag{2}$$

For ease of visualization, the subscript t has been eliminated.

This model has been a huge success in the finance field mainly because it is simple (results can be easily reported and explained) and intuitive (the rationale behind it is

[1]Investment strategies that involve two decisions taken in the following order: allocation (invest in sectors that are forecasted to outperform the market) and selection (among those sectors, select the stocks to invest in).

[2]The interaction effect has been included into the selection effect; for further reference: Laker, Damien. 2000. *What is this Thing Called Interaction?.* The Journal of Performance Measurement; vol. 5, no. 1, (Fall), pp. 43-57.

[3]$\overline{w}_i$: benchmark weight of asset i;
w_i: portfolio weight of asset i;
$\overline{r}_i$: return attained by asset i in the benchmark;
r_i: return attained by asset i in the portfolio;
R: portfolio return (single-period);
$\overline{R}$: benchmark return (single-period).

easy to follow). On the other hand, it has one main drawback: it cannot be effectively implemented into a multi-period timeframe because both the sum and the compounding of the single-period arithmetic returns are not equal to the multi-period arithmetic return[4]. Specifically:

$$R_T - \overline{R}_T \neq \sum_{t=1}^{T}(R_t - \overline{R}_t); \tag{6}$$

or:

$$R_T - \overline{R}_T = \sum_{t=1}^{T}(R_t - \overline{R}_t) + \epsilon_{A_r}. \tag{7}$$

Where ϵ_{A_r} is called residual: a variable that can be either positive or negative and it is the sum of all the cross products that arise in the calculation above[5].

The Brinson model decomposes the active returns and, as a consequence, it does not make sense to decompose an active return that appears to be unusable at a multi-period level.

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[4]For instance, for a three periods investment the arithmetic multi-period active return is equal to:

$$R_T - \overline{R}_T = (1 + R_1)(1 + R_2)(1 + R_3) - (1 + \overline{R}_1)(1 + \overline{R}_2)(1 + \overline{R}_3). \tag{3}$$

This is different from the sum of the single-period active returns (the arithmetic active return is based on the simple interest rationale; as a consequence, it is logical to sum the single-period active returns to calculate the multi-period active return):

$$R_T - \overline{R}_T \neq (R_1 - \overline{R}_1) + (R_2 - \overline{R}_2) + (R_3 - \overline{R}_3). \tag{4}$$

The multi-period active return is also different from the compounding of the single-period active returns (this is to demonstrate that also under the compounding method the equation does not hold):

$$R_T - \overline{R}_T \neq (1 + R_1 - \overline{R}_1)(1 + R_2 - \overline{R}_2)(1 + R_3 - \overline{R}_3) - 1. \tag{5}$$

[5]For the three-period investment of the preceding footnote:
The multi-period active return is:

$$R_T - \overline{R}_T \neq (1 + R_1)(1 + R_2)(1 + R_3) - (1 + \overline{R}_1)(1 + \overline{R}_2)(1 + \overline{R}_3). \tag{8}$$

Performing the calculations:

$$R_T - \overline{R}_T \neq (R_1 - \overline{R}_1) + (R_2 - \overline{R}_2) + (R_3 - \overline{R}_3) + R_1R_2 + R_1R_3 + R_2R_3 + R_1R_2R_3 - \overline{R}_1\overline{R}_2 - \overline{R}_1\overline{R}_3 - \overline{R}_2\overline{R}_3 - \overline{R}_1\overline{R}_2\overline{R}_3. \tag{9}$$

Where the residual is:

$$\epsilon = R_1R_2 + R_1R_3 + R_2R_3 + R_1R_2R_3 - \overline{R}_1\overline{R}_2 - \overline{R}_1\overline{R}_3 - \overline{R}_2\overline{R}_3 - \overline{R}_1\overline{R}_2\overline{R}_3. \tag{10}$$

As it is possible to see:

$$R_T - \overline{R}_T = \sum_{t=1}^{T}(R_t - \overline{R}_t) + \epsilon_{A_r}. \tag{11}$$

This equation explains the residual under the arithmetic approach. The residual under the geometric approach will be analyzed later.

Two paths has been followed to overcome this problem:

1. Continue with the arithmetic decomposition of the active return. The problem to solve when following this path is to allocate the residual among the single-period effects. This process is done by means of the so-called arithmetic linking algorithms: equations that split the residual among single-period effects [6] and that are based on the rationale of the simple interest rate (rate of returns are summed to and subtracted from each other to calculate the multi-period rate of return).

2. Shift to a geometric decomposition of the active return and then apply the Brinson model to the return thus calculated. The geometric active return is calculated as: $GV_t = \frac{1+R_t}{1+\overline{R}_t} - 1$. This can be effectively implemented into a multi-period context because the compounding of the single-period active returns equals the multi-period active return[7]. Therefore, this approach provides a multi-period active return that can be effectively decomposed. Nonetheless, the problems arise when the Brinson model - refined for a geometric use - is applied to this base: simply speaking, if the geometric active return is decomposed into allocation and selection decisions, it cannot be effectively recomposed. This problem, that will be deeply analyzed in the following paragraphs, is solved by the geometric linking algorithms[8].

At this point, we have to decide which path is preferable: arithmetic active return or geometric active return? The literature on the topic is still divided: some authors prefer the intuitiveness of the arithmetic active return, some others the preciseness of the geometric active return [9]. In our opinion, the question can be summarized by referring to a topic of

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[6]The most common arithmetic linking algorithms are:

- Cariño, David R. 1999. *Combining Attribution Effects Over Time.* The Journal of Performance Measurement; vol. 3, no. 4 (Summer), pp. 5-14.
- Frongello, Andrew Scott Bay. 2002. *Linking Single-period Attribution Results.* The Journal of Performance Measurement; vol. 6, no. 3 (Spring), pp. 1022.
- Menchero, Jose. 2004. *Multiperiod Arithmetic Attribution.* Financial Analysts Journal. vol. 60, no. 4, pp. 76-91.
- Mirabelli, Andre. 2001. *The Structure and Visualization of Performance Attribution.* The Journal of Performance Measurement; vol. 5, no. 2 (Winter), pp. 55-80.

[7]Continuing from the example above, for a three periods investment the geometric multi-period active return is equal to:

$$\frac{1+R_T}{1+\overline{R}_T} - 1 = \frac{(1+R_1)(1+R_2)(1+R_3)}{(1+\overline{R}_1)(1+\overline{R}_2)(1+\overline{R}_3)} - 1. \tag{12}$$

The compounding of the single-period geometric added values is:

$$\frac{1+R_1}{1+\overline{R}_2} \cdot \frac{1+R_2}{1+\overline{R}_2} \cdot \frac{1+R_3}{1+\overline{R}_3} - 1 = \frac{1+R_T}{1+\overline{R}_T} - 1. \tag{13}$$

[8]For a clear presentation of geometric and arithmetic linking algorithms: Mirabelli, Andre. 2001. *The Structure and Visualization of Performance Attribution.* The Journal of Performance Measurement; vol. 5, no. 2 (Winter), pp. 55-80.

[9]For further reference: Bacon, Carl. 2002. *Excess Returns - Arithmetic or Geometric?* The Journal of Performance Measurement; vol. 6, no. 3 (Spring), pp. 23-31.

the first class of corporate finance taken at the University: real interest rate calculation.

Real interest rate is intuitively calculated as:

$$\text{real interest rate} \cong \text{nominal interest rate - inflation rate;} \tag{14}$$

but is correctly computed as:

$$\text{(1+real interest rate)-1} = \frac{\text{(1+nominal interest rate)}}{\text{(1+inflation rate)}} - 1. \tag{15}$$

As a consequence, we recognize the intuitiveness and the simplicity of the former calculation (arithmetic active return) but we prefer the preciseness and the correct computation of the latter one (geometric active return). Preciseness and correct computation are a must when dealing with portfolios worth many tens of million dollars because a small percentage difference can lead to a significant misallocation of funds. Therefore, the article will now focus on the geometric active return: it will briefly explain its rationale and it will develop "the unified framework for performance and risk attribution", an approach that is aimed to solve the problems of the geometric method by looking at it from a different perspective.

3 Geometric performance attribution

This paragraph explains the path that we decided to follow and its problems.

Geometric performance attribution started to become popular during the second half of the 1990s. This method calculates and decomposes the active return of an investment differently from the arithmetic approach. As Menchero stated ≪ the relative performance is defined arithmetically in terms of a difference and geometrically in terms of a ratio ≫[10]. As a consequence, if the arithmetic method needs a subtraction to calculate the active return, the geometric method needs a ratio. This method presents no problem at a multi-period level: the compounding of the single-period active returns thus calculated correctly equals the multi-period active return.

The problem, though, arises when the Brinson model is applied to the single-period active return. In fact, once the performance of the period is decomposed among single-sector selection and allocation effects, it cannot be effectively recomposed to form the portfolio-level single-period active return. The problem can be better understood if mathematically described as follows.

The Brinson model applied to the geometric framework uses two ratios to decompose the geometric active return of the period. The first one is used to calculate allocation and selection effects at the portfolio-level (a level that expresses the aggregate of all the allocation and selection decisions taken during the analyzed period)[11]:

[10] Menchero, Jose. 2000. *An Optimized Approach to Linking Attribution Effects Over Time.* The Journal of Performance Measurement;. vol. 5. no. 1 (Fall), pp. 36-42.

[11] For ease of explanation, the subscript t has been eliminated since the calculations are always referred to a single-period timeframe.

$$Allocation\, portfolio - level \quad : \quad \frac{(1+\sum_i w_i \overline{r}_i)(1+\sum_i \overline{w}_i \overline{R})}{(1+\sum_i \overline{w}_i \overline{r}_i)(1+\sum_i w_i \overline{R})} - 1 = A_i^P; \tag{16}$$

$$Selection\, portfolio - level \quad : \quad \frac{(1+\sum_i w_i r_i)}{(1+\sum_i w_i \overline{r}_i)} - 1 = S_i^P. \tag{17}$$

Then, these two ratios are broken down to calculate the allocation and selection decisions taken for each sector of investment:

$$Allocation\, sector - level \quad : \quad \frac{(1+w_i \overline{r}_i)(1+\overline{w}_i \overline{R})}{(1+\overline{w}_i \overline{r}_i)(1+w_i \overline{R})} - 1 = A_i^0; \tag{18}$$

$$Selection\, sector - level \quad : \quad \frac{(1+w_i r_i)}{(1+w_i \overline{r}_i)} - 1 = S_i^0. \tag{19}$$

Once the sector-level decisions have been broken down, they need to be re-aggregated at the fund-level by using the compounding method (geometric approaches are hinged on the compounded interest rationale). As a consequence, the re-aggregated fund-level figures are calculated as follows:

$$Allocation\, re - aggregated \quad : \quad \prod_i \frac{(1+w_i \overline{r}_i)(1+\overline{w}_i \overline{R})}{(1+\overline{w}_i \overline{r}_i)(1+w_i \overline{R})} - 1 = A_i^{P0}; \tag{20}$$

$$Selection\, re - aggregated \quad : \quad \prod_i \frac{(1+w_i r_i)}{(1+w_i \overline{r}_i)} - 1 = S_i^{P0}. \tag{21}$$

It is possible to note that the portfolio-level and the re-aggregated figures have a slightly different expression. Taking for example the selection effect it is possible to see that the equation does not hold:

$$\frac{(1+\sum_i w_i r_i)}{(1+\sum_i w_i \overline{r}_i)} - 1 \neq \prod_i \frac{(1+w_i r_i)}{(1+w_i \overline{r}_i)} - 1; \tag{22}$$

$$S_i^P \neq S_i^{P0}. \tag{23}$$

Rewriting the equation above to make the equality hold:

$$\frac{(1+\sum_i w_i r_i)}{(1+\sum_i w_i \overline{r}_i)} - 1 = \prod_i \frac{(1+w_i r_i)}{(1+w_i \overline{r}_i)} - 1 + \epsilon_S; \tag{24}$$

$$S_i^P = S_i^{P0} + \epsilon_S. \tag{25}$$

ϵ_s is the residual created by the selection effect under the geometric approach[12] and, in order to make the equality hold, it needs to be attributed between the sector-level selection effects. The same happens for the allocation effect. This process must be performed because it is not possible to evaluate an investment decision by leaving outside a piece of return (especially if the portfolio is worth many tens of million dollars). This procedure can be performed by finding a driver to allocate the residual, a process performed by the geometric linking algorithms. Mathematically, geometric linking algorithms creates a term[13] named Γ_{im} that makes the equality hold by slightly modifying the sector-level effects:

$$\left(\prod_i \frac{(1+w_i\overline{r}_i)(1+\overline{w}_i\overline{R})}{(1+\overline{w}_i\overline{r}_i)(1+w_i\overline{R})}\Gamma_{iAllocation}\right)\left(\prod_i \frac{(1+w_ir_i)}{(1+w_i\overline{r}_i)}\Gamma_{iSelection}\right) = \frac{(1+\sum_i w_ir_i)}{(1+\sum_i w_i\overline{r}_i)} \tag{28}$$

As a result, the difficulty is to appropriately calculate the term Γ_{im} since this is the part that will modify each sector effect. The rationale used to calculate this term is the base where a linking algorithm is built upon.

There are many geometric linking algorithms in the literature but, in our opinion, the most prominent one is the "Optimized Geometric Attribution" laid out by Jose Menchero in 2005. This will be the benchmark used to test our approach. In the construction of this linking algorithm the author ≪[...] sought to distribute the residual in such a way that the attribution effects would deviate minimally from their pure form while taking special care to ensure that small attribution effects were not disproportionately affected≫[14]. In other words, the rationale to calculate the variable Γ_{im} is based on two criteria:

1. Make this variable as close to 1 as possible, so to not sharply modify the original attributes;

2. Modify each attribute proportionally to its size (in absolute values, bigger attributes need to be modified more compared to smaller attributes).

Menchero, by solving a constrained mathematical programming problem, obtained this solution[15]:

[12] Specifically, for a two-sector portfolio the calculations for the selection effect are:

$$\frac{(1+w_1r_1+w_2r_2)}{(1+w_1\overline{r}_1+w_2\overline{r}_2)} - 1 \neq \frac{(1+w_1r_1)(1+w_2r_2)}{(1+w_1\overline{r}_1)(1+w_2\overline{r}_2)} - 1. \tag{26}$$

As it is possible to note, the right branch of the equation gives rise to cross products that are not present on the left branch:

$$\frac{(1+w_1r_1+w_2r_2)}{(1+w_1\overline{r}_1+w_2\overline{r}_2)} - 1 \neq \frac{(1+w_1r_1+w_2r_2+w_1r_1w_2r_2)}{(1+w_1\overline{r}_1+w_2\overline{r}_2+w_1\overline{r}_1w_2\overline{r}_2)} - 1. \tag{27}$$

[13] Where i is the sector and m the effect under consideration.

[14] Menchero, Jose. 2005. *Optimized Geometric Attribution.* Financial Analysts Journal; vol. 61, no. 4, pp. 60-69.

[15] For any further reference refer to: Menchero, Jose. 2005. *Optimized Geometric Attribution.* Financial Analysts Journal; vol. 61, no. 4, pp. 60-69.

$$\prod_i (1+A_i^0)\Gamma_{Ai}(1+S_i^0)\Gamma_{Si} = \frac{1+R}{1+\overline{R}}. \tag{29}$$

Where:

- $\Gamma_{Ai} = e^{\ln^2(1+A_i^0)Q}$;
- $\Gamma_{Si} = e^{\ln^2(1+S_i^0)Q}$;
- $Q = \frac{\ln(1+R)-\ln(1+\overline{R})-\sum_i \ln(1+A_i^0)(1+S_i^0)}{\sum_i \ln^2(1+A_i^0)+\sum_i \ln^2(1+S_i^0)}$

In simple words, this linking algorithm calculates a variable Q that is constant for every sector effect. Then, the variable Q is multiplied by a term that includes the sector effect. In our opinion, this linking algorithm has advantages and disadvantages. The biggest advantage is that it clearly states that the modification of each sector effect is something undesirable that needs to be kept at its lowest level. We agree with this statement because when a sector attribute is modified it creates two disadvantages:

1. The sense of mistrust in the fund sponsor[16];
2. A potentially flawed result: modifying attributes either by adding or subtracting a small rate of return can cause the attribute to be even more flawed.

As a consequence, we agree with Menchero's rationale to least perturb the attributes and to modify bigger attributes more (the bigger the absolute size of the attribute, the bigger the modification that this attribute can bear without significantly modifying its meaning).

Nevertheless, we disagree with Menchero's linking algorithm in two points:

1. It does not take into account the residual created by allocation and selection decisions considered separately. Specifically, portfolio-level attributes do not create problems: their compounding correctly adds up to the active return of the period. The problem arises when these attributes are broken down at the sector-level because the compounding of sector-level effects does not equal the portfolio-level attributes. As a consequence, there are two residuals that need to be allocated: one for the allocation effect and the other for the selection effect. Menchero's does not consider this distinction: it re-allocates the residual created at a total portfolio level indistinctly among allocation and selection decisions. As a result, the equality still holds but its residual distribution could be ameliorated.
2. Return is only one part of the story, risk is absent from this linking algorithm. In our opinion, risk must be included into the economic driver responsible for allocating the residual because it is an inseparable part of the investment decision. A big rate of return can become fragile if it is compared to the risk taken to attain it. As a consequence, the size of an investment decision should be represented by a risk-return adjusted measure.

[16]The people to whom the analysis is meant for.

By improving these two points it is possible to arrive at a residual allocation that is more precise and more consistent with the investment process followed (a process that takes into consideration the expected return as well as the expected risk). The first point could be resolved with the tools that have been highlighted so far. As to the second point, it needs a risk-decomposition framework to be solved. A framework that has been recently developed in the literature and that will be explained in the next paragraph.

4 Risk attribution using the Brinson model

This paragraph investigates the application of the Brinson model for risk analysis purposes. Usually risk and return are strictly correlated but ≪ a common practice in asset management today is to use one model for attributing portfolio returns and to use an entirely different model for attributing risk. For instance, the active return of a portfolio is often decomposed into the allocation and selection effects by using [the Brinson Model]. The active risk of that portfolio, however, is typically attributed to a set of factors within a fundamental factor model. This inconsistency obscures the intimate link between the sources of risk and return ≫[17]. This is the reason why the Brinson model has been adapted for risk attribution purposes. In this way it is possible to analyze the investment decisions using a consistent framework.

On the background of this framework there is the Harry Markowitz's Modern Portfolio Theory, a widely known and used model that owes its success to the intuitiveness of its results. We acknowledge that there are more sophisticated techniques for risk analysis but not all of those can be applied to the Brinson model because this needs risk measures that can be subdivided among the sectors of investment. In this case variance and covariance prove to be useful because they can be easily decomposed. Therefore, this paragraph analyses the *X-Sigma-Rho Formula*[18]: an approach that adapts the MPT to the Brinson model. Nonetheless, this model can be applied only to arithmetic active returns because variance and covariance are decomposed by means of additions, the same operations that are used in the arithmetic Brinson model. We will adapt this model to the geometric framework in the next paragraph where we will issue also some improvements to it.

The *X-Sigma-Rho* measures the risk of the investment strategy as the standard deviation of the active return. Then, the standard deviation is broken down among allocation and selection attributes by using the usual relationship between standard deviation of the sector effect multiplied by the correlation that this effect has with the active return of the portfolio[19]:

$$\sigma(R_A) = \sum_i (\sigma(A_i)\rho(A_i, R_A) + \sigma(S_i)\rho(S_i, R_A)). \tag{30}$$

[17] Menchero, Jose. Poduri, Vijay. 2008. *Custom Factor Attribution.* Financial Analysts Journal; vol. 64, no. 2, pp. 81-92.

[18] Menchero, Jose. Davis, Ben. 2011. *Risk Contribution Is Exposure Times Volatility Times Correlation: Decomposing Risk Using the X-Sigma-Rho Formula.* The Journal of Portfolio Management; Winter, vol. 37, no. 2, pp. 97-106.

[19] R_A is the active return of the portfolio, σ the standard deviation and ρ the correlation.

The expression $\sigma(A_i)\rho(A_i, R_A)$ represents the risk faced by the allocation effect and $\sigma(S_i)\rho(S_i, R_A)$ is the risk faced by the selection effect.

The formula above has been modified by Menchero and Davis as:

$$\sigma(R_A) = \sum_i ((w_i - \overline{w}_i)\sigma((\overline{r}_i - \overline{R}))\rho((\overline{r}_i - \overline{R}), R_A) + (w)_i\sigma((r_i - \overline{r}_i))\rho((r_i - \overline{r_i}), R_A)) \quad (31)$$

As it is possible to see, the weights of allocation and selection effects have been taken out from the variance and covariance calculations. By performing this operation, the formula openly highlights the three components of risk: exposure (the weights), volatility and correlation. On the other hand, mathematically this operation can be performed only if the weights remain constant throughout the period[20]. This happens rarely[21] because different securities have different rate of returns and therefore their weights change[22] making the results of this formula are slightly flawed. In conclusion, we prefer the former expression of the standard deviation decomposition because it is more precise and leads to a more correct risk evaluation.

Now risk and return must be framed in a unified metric: the information ratio adapted to the Brinson model analysis.

The information ratio is usually defined as the average active return over the standard deviation of the active return[23]:

$$IR = \frac{\overline{R_A}}{\sigma(R_A)}. \quad (32)$$

Analyzing this formula it is possible to note that standard deviation is always positive (better, nonnegative) by construction, hence the active return determines the sign of the IR.

In the case of the Brinson model, the ratio laid out above is modified by changing the average excess return with the portfolio-level actual excess return at the date of evaluation:

$$IR = \frac{R_A}{\sigma(R_A)}. \quad (33)$$

Now it is possible to decompose the active return R_A as a sum of the attribution effects Q_m[24]. Consequently, the IR becomes:

$$IR = \frac{\sum_m Q_m}{\sigma(R_A)}. \quad (34)$$

[20]Specifically, $Var(aB) = a \times Var(B)$ only if a is constant. The same happens for covariance.

[21]Only if the benchmark and the portfolio are rebalanced at the same time of the rate of return measurement.

[22]For a more detailed explanation of changes in weights see: Laker, Damien. 2003. *Benchmark Rebalancing Calculations.* The Journal of Performance Measurement; vol. 7, no. 3 (Spring) pp. 8-23.

[23]Where $\overline{R_A}$ is the mean average of the active returns for the period under consideration.

[24]A decomposition firstly performed by Xiang, George. 2006. *Risk Decomposition and its Use in Portfolio Analysis.* The Journal of Performance Measurement; vol. 9, no. 2, pp. 26-32.

As it has been shown before, $\sigma(R_A)$ can be decomposed into: $\sigma(Q_m)\rho(Q_m, R_A)$. Therefore the ratio above can be expressed as:

$$IR = \frac{\sum_m Q_m}{\sum_m \sigma(Q_m)\rho(Q_m, R_A)}. \tag{35}$$

Now, this ratio can be broken into the different components of the Brinson model: allocation and selection effects of each sector. In order to achieve this result, the information ratio can be split across these different terms as:

$$IR(Q_m) = \frac{Q_m}{\sigma(Q_m)\rho(Q_m, R_A)}. \tag{36}$$

This ratio is the so-called "component information ratio"; Q_m is the effect under consideration and $\sigma(Q_m)\rho(Q_m, R_A)$ is the percentage risk contribution of the effect to the total risk $\sigma(R_A)$. This ratio has been further subdivided into:

$$IR(Q_m) = \frac{Q_m}{\sigma(Q_m)}\left(\frac{1}{\rho(Q_m, R_A)}\right); \tag{37}$$

where $\frac{Q_m}{\sigma(Q_m)}$ is named "stand-alone information ratio". This distinction will become useful in the following paragraph.

The component information ratios are summed up to the portfolio information ratio using a weighted average. In fact, the component information ratios need to be weighted for a risk contribution parameter named u_m[25]:

$$u_m = \frac{\sigma(Q_m)\rho(Q_m, R_A)}{\sigma(R_A)}. \tag{38}$$

This risk decomposition is then applied to every attribute of the Brinson model. It is useful in portfolio analysis because it places side by side return achieved and risk faced, providing a comprehensive summary of how effective a single investment decision has been.

In conclusion, the inclusion of the risk attribution field into the Brinson model gives a comprehensive indicator of how effective an investment strategy has been. This tool, though, could be further refined by adapting it to a geometric rate of return and by modifying the calculation of the information ratio. These steps will be performed in the next paragraph.

5 A unified framework for performance and risk attribution

The previous paragraphs issued all the elements necessary to develop the *unified framework for performance and risk attribution*: a geometric approach that allocates the residual using a risk-return adjusted driver. The outcome of this linking algorithm is a more sincere and transparent output where residual allocation is performed consistently with the investment process (where both the risk and the return of an asset are taken into account).

[25] $\sum_m u_m = 1$

In order to develop this approach, we need to: adapt the risk attribution process to a geometric framework, create a useful driver to attribute the residual and then issue the modified Brinson attributes. As to the first step, briefly recall that the X-Sigma-Rho framework decomposes the standard deviation of the active return into a weighted sum of allocation and selection effects:

$$\sigma(R_A) = \sum_i (\sigma(A_i)\rho(A_i, R_A) + \sigma(S_i)\rho(S_i, R_A)). \tag{39}$$

This formula can be applied only to an arithmetical active return ($R_A = \sum_i A_i + S_i$) because standard deviation is decomposed by means of additions. The geometric approach, instead, calculates the active return[26] through a compounding calculation:

$$(1 + R_G^0) = \prod_i (1 + A_i^0)(1 + S_i^0). \tag{40}$$

Recall that R_G^0 (the compounding of unadjusted allocation and selection effects) is slightly different from GV_t (the geometric active return): the difference, called residual, is what we are trying to attribute.

Therefore, the geometric approach needs to be adapted to an additive form in order to allow the above-stated risk decomposition. This process can be performed by using logarithms and, as a consequence, by moving to a continuously compounded rate of return[27]:

$$\ln(1 + R_G^0) = \sum_i (1 + A_i^0)(1 + S_i^0). \tag{41}$$

In this way attributes can be summed to calculate the total active return of the period. The continuously compounded returns will be converted into their normal form by means of the exponential function at the end of the process.

Now that the unadjusted attributes have been converted into an additive form, it is possible to allocate the residual among them by using the following equation:

$$\ln(1 + GV_t) = \sum_i (\ln(1 + A_i)\Gamma_{iAllocation} + \ln(1 + S_i)\Gamma_{iSelection}). \tag{42}$$

In order to perform this task, we need to find the drivers Γ_i for allocation and selection effects. We want to use an economic driver that takes into account the return achieved as well as the risk taken: the absolute size of the Information Ratio of each decision. The rationale behind this process is that the residuals created by allocation and selection effects considered separately need to be allocated proportionally to the size of each decision taken. Specifically, every time a residual is allocated, it modifies the size of each decision: it slightly alters the return achieved and its standard deviation. As a consequence, the bigger the size of an investment decision the more residual it can bear without significantly altering its meaning. To us, its meaning is given by the Information Ratio of the decision: a metric

[26] R_G^0 is the geometric active return calculated as the compounding of unadjusted allocation and selection effects.

[27] It is possible to use logarithms because: $1 + R_G^0; (1 + A_i^0); (1 + A_i^0) > 0$

that takes into account both its risk and its return. This must be the point of reference to evaluate an investment decision: attributing the residual by taking into account only the return achieved is a short-sighted vision of the investment because risk is absent from this apportionment. Moreover, its size is the absolute value of the IR: the driver used to allocate the residuals. This because if an IR is extremely high or extremely low it gives a clear representation of the investment decision taken: profitable or unprofitable. On the other hand, if an IR is close to zero it does not give an immediate meaning of what the investment decision has been: this is the case to more deeply investigate that decision. If a decision needs to be further investigated, it is better to leave it without a big piece of residual because the residual modifies its meaning. To perform this process we need an IR that clearly depicts the investment decision. In our opinion, the component IR laid out in the previous paragraph is not suitable to every market condition. The component information ratio is expressed as: $IR(Q_m) = \frac{Q_m}{\sigma(Q_m)\rho(Q_m,R_A)}$. Moreover, recall that a positive IR means a profitable investment decision. This is not always the case with the component IR because, analyzing the ratio, it is possible to see that the denominator of the function comprises the standard deviation of the component (that is always positive) multiplied by the correlation between the component returns with the active returns (a quantity that can also be negative). If the correlation is negative, the interpretation of the IR becomes not intuitive because a negative return divided by a negative correlation gives rise to a positive component information ratio. A positive IR is synonym of a profitable investment decision. But the decision has not been profitable since it attained a negative return. As a result, the meaning that the information ratio gives in this situation can be misunderstood. Consequently, we suggest to use the stand-alone rather than the component IR. In fact, the stand-alone IR provides a risk evaluation framework that is robust to every market condition, including a negative correlation. Stand-alone IR[28] is the return of the effect under consideration divided by its standard deviation:

$$IR_S = \frac{Q_m}{\sigma(Q_m)}. \tag{43}$$

As it is possible to note, this ratio is coherent with the original IR formula where the sign of the ratio is given only by the numerator. The portfolio IR is calculated as a weighted sum of all the stand-alone IR multiplied by the factor[29] $u_{mS} = \frac{\sigma(Q_m)}{\sigma(R_A)}$. As it is possible to note, the equality still holds and the IR interpretation is now coherent with its original definition.

After these changes it is possible to allocate the residual (recall that we are still using continuously compounded returns). For example, taking into consideration the allocation effect, we know that[30]:

$$\sum_i \ln(1 + A_i^0) + \varepsilon_A = \ln(1 + A_i^P). \tag{44}$$

[28] Where S stands for stand-alone and m is the investment decision under consideration.
[29] The factor u_{mS} is not a weight anymore because $\sum_m u_{mS} \neq 1$.
[30] ε_A is the residual created by the allocation effect.

ε_A will be embedded into the sector effects by means of the variable $\Gamma_{iAllocation}$. This variable, given the process laid out before, is calculated for each effect as [31]

$$\Gamma_{iAllocation} = \frac{|IR_S(All(i))|}{|IR_S(All(1))| + |IR_S(All(2))| + ... + |IR_S(All(n))|}(\varepsilon_A). \tag{45}$$

For ease of visualization, the quantity $\frac{|IR_S(All(i))|}{|IR_S(All(1))|+|IR_S(All(2))|+...+|IR_S(All(n))|}$ is named β_{Ai}: the residual bearing percentage for the allocation effect of sector i. The same process happens for the selection effect [32].

As it is possible to see, it is the absolute size of each stand-alone IR that gives the residual bearing percentage of each sector. This is coherent with the process laid out before.

Now that the continuously compounded returns have been modified, it is possible to move to the percentage returns using the exponential function. Therefore, our final formula is:

$$\prod_i (1 + A_i^0)\Gamma_{Ai}(1 + S_i^0)\Gamma_{Si} = \frac{1 + R}{1 + \overline{R}}. \tag{46}$$

Where:

- $\Gamma_{Ai} = e^{\beta_{Ai}(\epsilon_A)}$;
- $\Gamma_{Si} = e^{\beta_{Si}(\epsilon_S)}$;

As it is possible to see this approach correctly satisfies its premises by creating a residual attribution that takes into account the return achieved as well as the risk taken.

This geometric linking algorithm now needs to be compared to Menchero's optimized geometric approach that in our opinion is one of the best linking algorithms presented in the literature and, therefore, a good benchmark to test our approach.

Briefly recall that Menchero's linking algorithm is:

$$\prod_i (1 + A_i^0)\Gamma_{Ai}(1 + S_i^0)\Gamma_{Si} = \frac{1 + R}{1 + \overline{R}}. \tag{47}$$

Where:

- $\Gamma_{Ai} = e^{\ln^2(1+A_i^0)Q}$;
- $\Gamma_{Si} = e^{\ln^2(1+S_i^0)Q}$;
- $Q = \frac{(\ln(1+R)-\ln(1+\overline{R})-\sum_i \ln(1+A_i^0)(1+S_i^0)}{\sum_i \ln^2(1+A_i^0)+\sum_i \ln^2(1+S_i^0)}$.

[31] $|IR_S(All(1))|$, for example, is the absolute size of the stand-alone IR of the allocation decision of sector 1.

[32] β_{Si} is the residual bearing percentage for the selection effect of sector i and ϵ_S its residual.

At a first glance, it is possible to note that our linking algorithm has a more simple and straightforward expression. This is an advantage when the analysis is reported to the fund sponsor: the simpler the linking algorithm expression, the easier the comprehension of the rationale behind it and the smaller the sense of mathematical manipulation perceived by the fund sponsor. Moreover, we find that our approach is more consistent with the investment theory used. Specifically, the Modern Portfolio Theory uses a return/risk perspective as a point of reference to evaluate the effectiveness of each investment decision. We view the investment process from the same angle and, as a consequence, the depiction of the investment decisions is more coherent with the process followed. Lastly, our approach correctly distinguishes between the residuals created by allocation and selection decisions considered separately. Specifically, Menchero attributes the residual created at the fund-level indistinctively among the sector effects; our linking algorithm distinguishes and keeps separate the residuals created by the allocation and selection effects.

All these differences stem from the different standpoints used to observe the underlying phenomenon: Menchero sees it from a return-only perspective; we see it from a return-risk perspective. Which one is better? Nobody can know it with certainty because both points of view have a certain level of subjectivity. The implementation of one approach rather than another one is a decision that strictly correlates with the opinions that the fund sponsor has with regard to this phenomenon.

Moreover, we found useful also the modification of the component IR into the stand-alone IR because, in some circumstances, provides a better depiction of the underlying investment decision.

In conclusion, the development of our approach or Menchero's is a weighted average of the differences shown above. The weights placed are a decision that is up to the fund sponsor only.

6 Conclusion

The unified framework for performance and risk attribution is a comprehensive method that analyzes the investment decisions by taking into account the return achieved as well as the risk taken. This perspective provides a more sincere residual allocation that is more coherent with the investment theory used. Moreover, the modification of some of the risk tools presented in the literature makes the risk evaluation process robust to every condition. These improvements benefit the analysis in three aspects: a more clear and easily reportable analysis, a more sincere output and a more coherent residual allocation. These elements will benefit both the fund sponsor as well as the investment manager.

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