A.M.A.S.E.S.

ASSOCIAZIONE PER LA MATEMATICA APPLICATA
ALLE SCIENZE ECONOMICHE E SOCIALI

ATTI DEL VENTITREESIMO CONVEGNO A.M.A.S.E.S.

Rende, 8-11 Settembre 1999

MERTON-LIKE THEORETICAL FRAME FOR FRACTIONAL BROWNIAN MOTION IN FINANCE

Marco CORAZZA *(e-mails:* ***corazza@unive.it****)*
Department of Applied Mathematics - University "Ca' Foscari" of Venice

1. Introduction

Generally, the behaviours of real financial asset returns are characterized by features which differ from the ones theoretically stated by the classical approach. Currently, among these empirical features, one of the most investigated is the presence of long-run dependence inside the asset returns (for example, see Poterba and Summers [1988] and Lo [1991]). From a distributional point of view, this memory can be modelled by the so-called *fractional Brownian* (fB) motion which is a Gaussian stochastic process whose increments are (long-term) dependent with each other.

Starting from these remarks, in this work we propose a Merton-like system of economic-financial assumptions on the dynamical behaviour of financial asset price by which it is possible to deduce the consistency between the fB motion and the discrete-time trading. Moreover, we also prove the "convergence" of the fB motion to the standard Brownian (sB) one when the discrete-time trading tends to the continuous-time one.

Our work is organized as follows. Firstly, we introduce the fB motion; then, we propose our Merton-like system of economic-financial assumptions, and, finally, we give the theoretical results we deduce from the previously stated axiomatical system.

2. FB motion

The fB motion, introduced by Mandelbrot and Van Ness [1968], is an almost everywhere continuous Gaussian stochastic process of index $H \in]0,1[$, $\{B_H(t), t \geq 0\}$, such that $B_H(0) = 0$ with probability 1 and $B_H(t_2) - B_H(t_1) \sim N\left(0, \sigma^{2H}(t_2 - t_1)^{2H}\right)$, with $0 \leq t_1 < t_2 < +\infty$ and $\sigma > 0$ (for more details, see Beran [1994]). If $H \neq 0.5$, the increments are stationary but long-term correlated with each other, and the fB motion is not a semi-martingale and, consequently, that there does not exist an equivalent martingale measure (for more details, see Rogers [1995] and Kopp [1995]).

3. The Merton-like axiomatical system

In this section we state a system of assumptions, quite inspired by the one proposed by Merton [1982], [1990], able to provide a theoretical frame for the fB motion in finance.

At first, in order to develop our axiomatical system, we introduce a bit of formalisms. Let the strictly positive real number h denote the investor uniperiodal time interval, i.e. the minimum time length between two successive transactions, $t_0 = 0$, and $t_n = T = nh < +\infty$, with $n \in \mathbf{N} \setminus \{0\}$, denote, respectively, the starting and the final time moment of the investor intertemporal time interval, and let $X(k)$, with $k=0, \ldots, n$, denote the financial asset price, or its logarithmical transformation, at time k. In general, for such random variables we can reasonably assume the following structure

$$X(k) - X(k-1) = E_{k-1}\{X(k) - X(k-1)\} + \varepsilon(k),\ k = 1, \ldots, n,$$

where $E_{k-1}\{\}$ is the expectation operator conditional on the relevant information available at time $k-1$ or before.

Starting from this background, we propose our Merton-like system of economic-financial assumptions.

<u>Assumption A.1</u> - *For each time period $[0,T]$ there exists a strictly positive real number $A_2 < +\infty$, independent on n, such that*

$$\sum_{k=1}^{n} Var\{X(k) - X(k-1)\} \leq A_2$$

where $Var\{\cdot\}$ is the variance operator.

<u>Assumption A.2</u> - *For each time period $[0,T]$ there exists a real number $A_3 \in]0,1]$, independent on n, such that*

$$A_3 \leq \frac{Var\{X(k) - X(k-1)\}}{\max_{k \in \{1,\ldots,n\}}\{Var\{X(k) - X(k-1)\}\}},$$

$$k = 1, \ldots, n, \text{ and } \max_{k \in \{1,\ldots,n\}}\{Var\{X(k) - X(k-1)\}\} > 0$$

where $\max_{k \in \{\}}\{\}$ is the maximum operator.

From the latter assumption it is possible to prove the existence of a strictly positive lower bound for the sum of the risks associated with the price variation over the n time period $[k, k-1]$, lower bound which is independent on the number of transactions.

<u>Proposition P.1</u> - *If assumption A.2 holds, then, for each time period $[0,T]$, there exists a strictly positive real number $A_1 < +\infty$, independent on n, such that*

$$A_1 \leq \sum_{k=1}^{n} Var\{X(k) - X(k-1)\}.$$

Concluding, notice that we jointly consider both the anticipable part of the price variation and the unanticipable one.

4. Dependence in asset returns: the results

The fact of dealing with dependence in a discrete-time financial market is mainly due to our aim to develop theoretical results in a more "realistic" environment. With such a spirit, we give our results, which are deduced using, besides the axiomatical system previously stated, also assumptions on the form of the variance of the (unknown) underlying stochastic process, and no other information.

<u>Theorem T.1</u> - *If assumptions A.1 and A.2 hold and if* $Var\{X(k)-X(k-1)\}$, *with k=1, ..., n, is proportional to* h^{2H}, *with proportionality factor* $\sigma \in [(A_1 A_3)/T, A_2/(TA_3)]$, *then H belongs to a proper neighbourhood of 0.5.*

As theorem T.1 states that *H* belongs to a suitable neighbourhood of 0.5, it shows that the proposed theoretical framework is "powerful" enough to ensure the consistency between the fB motion and the discrete-time trading without having to exclude the classical sB motion case. Moreover, this theorem leaves open the possibility that *H* can vary over time.

The next corollary provides the necessary conditions so that the variance of the underlying stochastic process moves like the one of the sB motion.

<u>Corollary C.1</u> - *If assumptions A.1 and A.2 hold, if* $Var\{X(k)-X(k-1)\}$, *with k=1, ..., n, is proportional to* h^{2H}, *with* $h \neq 1$ *and with proportionality factor* $\sigma > 0$, *if* $A_1 = (T\sigma)/A_3$ *and if* $A_2 = T\sigma A_3$, *then H=0.5.*

Finally, the next proposition states the "convergence" of the fB motion to the sB one when the discrete-time trading tends to the continuous-time one.

<u>Proposition P.2</u> - *If assumptions A.1 and A.2 hold, if* $Var\{X(k)-X(k-1)\}$, *with k=1, ..., n, is proportional to* h^{2H}, *with* $h \neq 1$ *and proportionality factor* $\sigma \in [(A_1 A_3)/T, A_2/(TA_3)]$, *and if h tends to* 0^+ *then H tends to 0.5.*

From a qualitative point of view, proposition P.2 suggests that the dependence inside the asset returns tends to vanish when these returns are realized by an higher and higher number of transactions. Such a behaviour finds some empirical check in the analysis of high-frequency data (for more details, see Evertsz [1995] and Evertsz and Berkner [1995]).

5. References

[1] J. Beran, *Statistics for Long-memory Processes*, Chapman & Hall, 1994.
[2] G.G. Booth , F.R. Kaen and P.E. Koveos, "R/S Analysis of Foreign Exchange Rates under Two International Monetary Regimes", *Journal of Monetary Economics*, 10, 1982, 407-415
[3] C.J.G. Evertsz, "Self-similarity of High-frequency USD-DEM Exchange Rates". In: *Proceedings of the First International Conference on High Frequency Data in Finance*, Zurich (Switzerland), 1995.
[4] C.J.G. Evertsz and K. Berkner, "Large Deviation and Self-similarity Analysis of Graphs: DAX Stock Prices", *Chaos, Solitons & Fractals*, 6, 1995, 121-130.
[5] E. Kopp, "Fractional Brownian Motion and Arbitrage", *mimeo*, University of Hull (England), 1995.
[6] A.W. Lo, "Long-term Memory in Stock Market Prices", *Econometrica*, 59(5), 1991, 1279-1313.
[7] B.B Mandelbrot and J.W. Van Ness, "Fractional Brownian Motions, Fractional Noises and Applications", *SIAM Review*, 10(4), 1968, 422-437.
[8] R.C. Merton, "On the Mathematics and Economics Assumptions of Continuous-Time Models". In: Sharpe W.F. and Cootner C.M. (eds.), *Financial Economics: Essays in Honor of Paul Cootner*, Prentice Hall, 1982.
[9] R.C. Merton, *Continuous-time Finance*, Basil Blackwell, 1990.
[10] J.M. Poterba and L.H. Summers, "Mean Reversion in Stock Prices. Evidence and Implications", *Journal of Financial Economics*, 22, 1988, 27-59.
[11] L.C.G. Rogers, "Arbitrage with Fractional Brownian Motion", *mimeo*, University of Bath (England), 1995.

Finito di stampare nel mese di luglio 1999
presso la Tipografia
"La Grafica Commerciale" di Mario Tocci
Cosenza